

Exercise 4.2

Question 1:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$$

Answer

$$\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = \begin{vmatrix} x & a & x \\ y & b & y \\ z & c & z \end{vmatrix} + \begin{vmatrix} x & a & a \\ y & b & b \\ z & c & c \end{vmatrix} = 0 + 0 = 0$$

[Here, the two columns of the determinants are identical]

Question 2:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

Answer

$$\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + R_2$, we have:

$$\begin{aligned} \Delta &= \begin{vmatrix} a-c & b-a & c-b \\ b-c & c-a & a-b \\ -(a-c) & -(b-a) & -(c-b) \end{vmatrix} \\ &= - \begin{vmatrix} a-c & b-a & c-b \\ b-c & c-a & a-b \\ a-c & b-a & c-b \end{vmatrix} \end{aligned}$$

Here, the two rows R_1 and R_3 are identical.

$\therefore \Delta = 0$.

Question 3:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$$

Answer

$$\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = \begin{vmatrix} 2 & 7 & 63+2 \\ 3 & 8 & 72+3 \\ 5 & 9 & 81+5 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 7 & 63 \\ 3 & 8 & 72 \\ 5 & 9 & 81 \end{vmatrix} + \begin{vmatrix} 2 & 7 & 2 \\ 3 & 8 & 3 \\ 5 & 9 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 7 & 9(7) \\ 3 & 8 & 9(8) \\ 5 & 9 & 9(9) \end{vmatrix} + 0 \quad \text{[Two columns are identical]}$$

$$= 9 \begin{vmatrix} 2 & 7 & 7 \\ 3 & 8 & 8 \\ 5 & 9 & 9 \end{vmatrix}$$

$$= 0 \quad \text{[Two columns are identical]}$$

Question 4:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$$

Answer

$$\Delta = \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$$

By applying $C_3 \rightarrow C_3 + C_2$, we have:

$$\Delta = \begin{vmatrix} 1 & bc & ab+bc+ca \\ 1 & ca & ab+bc+ca \\ 1 & ab & ab+bc+ca \end{vmatrix}$$

Here, two columns C_1 and C_3 are proportional.

$$\therefore \Delta = 0.$$

Question 5:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

Answer

$$\begin{aligned} \Delta &= \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} \\ &= \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a & p & x \end{vmatrix} + \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ b & q & y \end{vmatrix} \\ &= \Delta_1 + \Delta_2 \quad (\text{say}) \quad \dots(1) \end{aligned}$$

$$\text{Now, } \Delta_1 = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a & p & x \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_3$, we have:

$$\Delta_1 = \begin{vmatrix} b+c & q+r & y+z \\ c & r & z \\ a & p & x \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 - R_2$, we have:

$$\Delta_1 = \begin{vmatrix} b & q & y \\ c & r & z \\ a & p & x \end{vmatrix}$$

Applying $R_1 \leftrightarrow R_3$ and $R_2 \leftrightarrow R_3$, we have:

$$\Delta_1 = (-1)^2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \quad \dots(2)$$

$$\Delta_2 = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ b & q & y \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 - R_3$, we have:

$$\Delta_2 = \begin{vmatrix} c & r & z \\ c+a & r+p & z+x \\ b & q & y \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$, we have:

$$\Delta_2 = \begin{vmatrix} c & r & z \\ a & p & x \\ b & q & y \end{vmatrix}$$

Applying $R_1 \leftrightarrow R_2$ and $R_2 \leftrightarrow R_3$, we have:

$$\Delta_2 = (-1)^2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \quad \dots(3)$$

From (1), (2), and (3), we have:

$$\Delta = 2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

Hence, the given result is proved.

Question 6:

By using properties of determinants, show that:

$$\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$$

Answer

We have,

$$\Delta = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

Applying $R_1 \rightarrow cR_1$, we have:

$$\Delta = \frac{1}{c} \begin{vmatrix} 0 & ac & -bc \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 - bR_2$, we have:

$$\begin{aligned} \Delta &= \frac{1}{c} \begin{vmatrix} ab & ac & 0 \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} \\ &= \frac{a}{c} \begin{vmatrix} b & c & 0 \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} \end{aligned}$$

Here, the two rows R_1 and R_3 are identical.

$\therefore \Delta = 0$.

Question 7:

By using properties of determinants, show that:

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Answer

$$\begin{aligned} \Delta &= \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} \\ &= abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} \quad \left[\text{Taking out factors } a, b, c \text{ from } R_1, R_2, \text{ and } R_3 \right] \\ &= a^2 b^2 c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \quad \left[\text{Taking out factors } a, b, c \text{ from } C_1, C_2, \text{ and } C_3 \right] \end{aligned}$$

Applying $R_2 \rightarrow R_2 + R_1$ and $R_3 \rightarrow R_3 + R_1$, we have:

$$\begin{aligned} \Delta &= a^2 b^2 c^2 \begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix} \\ &= a^2 b^2 c^2 (-1) \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} \\ &= -a^2 b^2 c^2 (0 - 4) = 4a^2 b^2 c^2 \end{aligned}$$

Question 8:

By using properties of determinants, show that:

$$(i) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

$$(ii) \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

Answer

$$(i) \text{ Let } \Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

Applying $R_1 \rightarrow R_1 - R_3$ and $R_2 \rightarrow R_2 - R_3$, we have:

$$\Delta = \begin{vmatrix} 0 & a-c & a^2-c^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (c-a)(b-c) \begin{vmatrix} 0 & -1 & -a-c \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + R_2$, we have:

$$\Delta = (b-c)(c-a) \begin{vmatrix} 0 & 0 & -a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)(c-a) \begin{vmatrix} 0 & 0 & -1 \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

Expanding along C_1 , we have:

$$\Delta = (a-b)(b-c)(c-a) \begin{vmatrix} 0 & -1 \\ 1 & b+c \end{vmatrix} = (a-b)(b-c)(c-a)$$

Hence, the given result is proved.

(ii) Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$.

Applying $C_1 \rightarrow C_1 - C_3$ and $C_2 \rightarrow C_2 - C_3$, we have:

$$\begin{aligned} \Delta &= \begin{vmatrix} 0 & 0 & 1 \\ a-c & b-c & c \\ a^3-c^3 & b^3-c^3 & c^3 \end{vmatrix} \\ &= \begin{vmatrix} 0 & 0 & 1 \\ a-c & b-c & c \\ (a-c)(a^2+ac+c^2) & (b-c)(b^2+bc+c^2) & c^3 \end{vmatrix} \\ &= (c-a)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & c \\ -(a^2+ac+c^2) & (b^2+bc+c^2) & c^3 \end{vmatrix} \end{aligned}$$

Applying $C_1 \rightarrow C_1 + C_2$, we have:

$$\begin{aligned} \Delta &= (c-a)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ (b^2-a^2)+(bc-ac) & (b^2+bc+c^2) & c^3 \end{vmatrix} \\ &= (b-c)(c-a)(a-b) \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ -(a+b+c) & (b^2+bc+c^2) & c^3 \end{vmatrix} \\ &= (a-b)(b-c)(c-a)(a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ -1 & (b^2+bc+c^2) & c^3 \end{vmatrix} \end{aligned}$$

Expanding along C_1 , we have:

$$\begin{aligned} \Delta &= (a-b)(b-c)(c-a)(a+b+c)(-1) \begin{vmatrix} 0 & 1 \\ 1 & c \end{vmatrix} \\ &= (a-b)(b-c)(c-a)(a+b+c) \end{aligned}$$

Hence, the given result is proved.

Question 9:

By using properties of determinants, show that:

$$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy + yz + zx)$$

Answer

$$\text{Let } \Delta = \begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}.$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we have:

$$\begin{aligned} \Delta &= \begin{vmatrix} x & x^2 & yz \\ y-x & y^2-x^2 & zx-yz \\ z-x & z^2-x^2 & xy-yz \end{vmatrix} \\ &= \begin{vmatrix} x & x^2 & yz \\ -(x-y) & -(x-y)(x+y) & z(x-y) \\ (z-x) & (z-x)(z+x) & -y(z-x) \end{vmatrix} \\ &= (x-y)(z-x) \begin{vmatrix} x & x^2 & yz \\ -1 & -x-y & z \\ 1 & z+x & -y \end{vmatrix} \end{aligned}$$

Applying $R_3 \rightarrow R_3 + R_2$, we have:

$$\begin{aligned} \Delta &= (x-y)(z-x) \begin{vmatrix} x & x^2 & yz \\ -1 & -x-y & z \\ 0 & z-y & z-y \end{vmatrix} \\ &= (x-y)(z-x)(z-y) \begin{vmatrix} x & x^2 & yz \\ -1 & -x-y & z \\ 0 & 1 & 1 \end{vmatrix} \end{aligned}$$

Expanding along R_3 , we have:

$$\begin{aligned} \Delta &= [(x-y)(z-x)(z-y)] \left[(-1) \begin{vmatrix} x & yz \\ -1 & z \end{vmatrix} + 1 \begin{vmatrix} x & x^2 \\ -1 & -x-y \end{vmatrix} \right] \\ &= (x-y)(z-x)(z-y) [(-xz - yz) + (-x^2 - xy + x^2)] \\ &= -(x-y)(z-x)(z-y)(xy + yz + zx) \\ &= (x-y)(y-z)(z-x)(xy + yz + zx) \end{aligned}$$

Hence, the given result is proved.

Question 10:

By using properties of determinants, show that:

$$(i) \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$$

$$(ii) \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$$

Answer

$$(i) \Delta = \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + R_2 + R_3$, we have:

$$\begin{aligned} \Delta &= \begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} \\ &= (5x+4) \begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} \end{aligned}$$

Applying $C_2 \rightarrow C_2 - C_1$, $C_3 \rightarrow C_3 - C_1$, we have:

$$\Delta = (5x+4) \begin{vmatrix} 1 & 0 & 0 \\ 2x & -x+4 & 0 \\ 2x & 0 & -x+4 \end{vmatrix}$$

$$= (5x+4)(4-x)(4-x) \begin{vmatrix} 1 & 0 & 0 \\ 2x & 1 & 0 \\ 2x & 0 & 1 \end{vmatrix}$$

Expanding along C_3 , we have:

$$\Delta = (5x+4)(4-x)^2 \begin{vmatrix} 1 & 0 \\ 2x & 1 \end{vmatrix}$$

$$= (5x+4)(4-x)^2$$

Hence, the given result is proved.

$$(ii) \Delta = \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + R_2 + R_3$, we have:

$$\Delta = \begin{vmatrix} 3y+k & 3y+k & 3y+k \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

$$= (3y+k) \begin{vmatrix} 1 & 1 & 1 \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

Applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$, we have:

$$\Delta = (3y+k) \begin{vmatrix} 1 & 0 & 0 \\ y & k & 0 \\ y & 0 & k \end{vmatrix}$$

$$= k^2 (3y+k) \begin{vmatrix} 1 & 0 & 0 \\ y & 1 & 0 \\ y & 0 & 1 \end{vmatrix}$$

Expanding along C_3 , we have:

$$\Delta = k^2 (3y+k) \begin{vmatrix} 1 & 0 \\ y & 1 \end{vmatrix} = k^2 (3y+k)$$

Hence, the given result is proved.

Question 11:

By using properties of determinants, show that:

$$(i) \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

$$(ii) \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$$

Answer

$$(i) \Delta = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + R_2 + R_3$, we have:

$$\begin{aligned} \Delta &= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \\ &= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \end{aligned}$$

Applying $C_2 \rightarrow C_2 - C_1$, $C_3 \rightarrow C_3 - C_1$, we have:

$$\begin{aligned} \Delta &= (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix} \\ &= (a+b+c)^3 \begin{vmatrix} 1 & 0 & 0 \\ 2b & -1 & 0 \\ 2c & 0 & -1 \end{vmatrix} \end{aligned}$$

Expanding along C_3 , we have:

$$\Delta = (a+b+c)^3 (-1)(-1) = (a+b+c)^3$$

Hence, the given result is proved.

$$(ii) \Delta = \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$, we have:

$$\begin{aligned} \Delta &= \begin{vmatrix} 2(x+y+z) & x & y \\ 2(x+y+z) & y+z+2x & y \\ 2(x+y+z) & x & z+x+2y \end{vmatrix} \\ &= 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix} \end{aligned}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we have:

$$\begin{aligned} \Delta &= 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 0 & x+y+z & 0 \\ 0 & 0 & x+y+z \end{vmatrix} \\ &= 2(x+y+z)^3 \begin{vmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \end{aligned}$$

Expanding along R_3 , we have:

$$\Delta = 2(x+y+z)^3 (1)(1-0) = 2(x+y+z)^3$$

Hence, the given result is proved.

Question 12:

By using properties of determinants, show that:

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$$

Answer

$$\Delta = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + R_2 + R_3$, we have:

$$\begin{aligned} \Delta &= \begin{vmatrix} 1+x+x^2 & 1+x+x^2 & 1+x+x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} \\ &= (1+x+x^2) \begin{vmatrix} 1 & 1 & 1 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} \end{aligned}$$

Applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$, we have:

$$\begin{aligned} \Delta &= (1+x+x^2) \begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1-x^2 & x-x^2 \\ x & x^2-x & 1-x \end{vmatrix} \\ &= (1+x+x^2)(1-x)(1-x) \begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1+x & x \\ x & -x & 1 \end{vmatrix} \\ &= (1-x^3)(1-x) \begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1+x & x \\ x & -x & 1 \end{vmatrix} \end{aligned}$$

Expanding along R_1 , we have:

$$\begin{aligned} \Delta &= (1-x^3)(1-x)(1) \begin{vmatrix} 1+x & x \\ -x & 1 \end{vmatrix} \\ &= (1-x^3)(1-x)(1+x+x^2) \\ &= (1-x^3)(1-x^3) \\ &= (1-x^3)^2 \end{aligned}$$

Hence, the given result is proved.

Question 13:

By using properties of determinants, show that:

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$$

Answer

$$\Delta = \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + bR_3$ and $R_2 \rightarrow R_2 - aR_3$, we have:

$$\begin{aligned} \Delta &= \begin{vmatrix} 1+a^2+b^2 & 0 & -b(1+a^2+b^2) \\ 0 & 1+a^2+b^2 & a(1+a^2+b^2) \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} \\ &= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -b \\ 0 & 1 & a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} \end{aligned}$$

Expanding along R_1 , we have:

$$\begin{aligned} \Delta &= (1+a^2+b^2)^2 \left[(1) \begin{vmatrix} a & 1-a^2-b^2 \\ -2a & 1-a^2-b^2 \end{vmatrix} - b \begin{vmatrix} 0 & 1 \\ 2b & -2a \end{vmatrix} \right] \\ &= (1+a^2+b^2)^2 [1-a^2-b^2+2a^2-b(-2b)] \\ &= (1+a^2+b^2)^2 (1+a^2+b^2) \\ &= (1+a^2+b^2)^3 \end{aligned}$$

Question 14:

By using properties of determinants, show that:

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$$

Answer

$$\Delta = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$$

Taking out common factors a , b , and c from R_1 , R_2 , and R_3 respectively, we have:

$$\Delta = abc \begin{vmatrix} a+\frac{1}{a} & b & c \\ a & b+\frac{1}{b} & c \\ a & b & c+\frac{1}{c} \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we have:

$$\Delta = abc \begin{vmatrix} a+\frac{1}{a} & b & c \\ -\frac{1}{a} & \frac{1}{b} & 0 \\ -\frac{1}{a} & 0 & \frac{1}{c} \end{vmatrix}$$

Applying $C_1 \rightarrow aC_1$, $C_2 \rightarrow bC_2$, and $C_3 \rightarrow cC_3$, we have:

$$\begin{aligned} \Delta &= abc \times \frac{1}{abc} \begin{vmatrix} a^2+1 & b^2 & c^2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \\ &= \begin{vmatrix} a^2+1 & b^2 & c^2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \end{aligned}$$

Expanding along R_3 , we have:

$$\begin{aligned}\Delta &= -1 \begin{vmatrix} b^2 & c^2 \\ 1 & 0 \end{vmatrix} + 1 \begin{vmatrix} a^2 + 1 & b^2 \\ -1 & 1 \end{vmatrix} \\ &= -1(-c^2) + (a^2 + 1 + b^2) = 1 + a^2 + b^2 + c^2\end{aligned}$$

Hence, the given result is proved.

Question 15:

Choose the correct answer.

Let A be a square matrix of order 3×3 , then $|kA|$ is equal to

A. $k|A|$ **B.** $k^2|A|$ **C.** $k^3|A|$ **D.** $3k|A|$

Answer

Answer: C

A is a square matrix of order 3×3 .

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

$$\text{Then, } kA = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \\ ka_3 & kb_3 & kc_3 \end{bmatrix}.$$

$$\therefore |kA| = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \\ ka_3 & kb_3 & kc_3 \end{vmatrix}$$

$$= k^3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(Taking out common factors k from each row)

$$= k^3 |A|$$

$$\therefore |kA| = k^3 |A|$$

Hence, the correct answer is C.

Question 16:

Which of the following is correct?

- A.** Determinant is a square matrix.
- B.** Determinant is a number associated to a matrix.
- C.** Determinant is a number associated to a square matrix.
- D.** None of these

Answer

Answer: C

We know that to every square matrix, $A = [a_{ij}]$ of order n . We can associate a number

called the determinant of square matrix A , where $a_{ij} = (i, j)^{\text{th}}$ element of A .

Thus, the determinant is a number associated to a square matrix.

Hence, the correct answer is C.