

Exercise 2.2**Question 1:**

$$3 \sin^{-1} x = \sin^{-1} (3x - 4x^3), x \in \left[-\frac{1}{2}, \frac{1}{2} \right]$$

Prove

Answer

$$3 \sin^{-1} x = \sin^{-1} (3x - 4x^3), x \in \left[-\frac{1}{2}, \frac{1}{2} \right]$$

To prove:

Let $x = \sin \theta$. Then, $\sin^{-1} x = \theta$.

We have,

$$\begin{aligned} \text{R.H.S.} &= \sin^{-1} (3x - 4x^3) = \sin^{-1} (3 \sin \theta - 4 \sin^3 \theta) \\ &= \sin^{-1} (\sin 3\theta) \\ &= 3\theta \\ &= 3 \sin^{-1} x \\ &= \text{L.H.S.} \end{aligned}$$

Question 2:

$$3 \cos^{-1} x = \cos^{-1} (4x^3 - 3x), x \in \left[\frac{1}{2}, 1 \right]$$

Prove

Answer

$$3 \cos^{-1} x = \cos^{-1} (4x^3 - 3x), x \in \left[\frac{1}{2}, 1 \right]$$

To prove:

Let $x = \cos \theta$. Then, $\cos^{-1} x = \theta$.

We have,

$$\begin{aligned}
 \text{R.H.S.} &= \cos^{-1}(4x^3 - 3x) \\
 &= \cos^{-1}(4\cos^3\theta - 3\cos\theta) \\
 &= \cos^{-1}(\cos 3\theta) \\
 &= 3\theta \\
 &= 3\cos^{-1}x \\
 &= \text{L.H.S.}
 \end{aligned}$$

Question 3:

Prove $\tan^{-1} \frac{2}{11} + \tan^{-1} \frac{7}{24} = \tan^{-1} \frac{1}{2}$

Answer

To prove: $\tan^{-1} \frac{2}{11} + \tan^{-1} \frac{7}{24} = \tan^{-1} \frac{1}{2}$

$$\text{L.H.S.} = \tan^{-1} \frac{2}{11} + \tan^{-1} \frac{7}{24}$$

$$= \tan^{-1} \frac{\frac{2}{11} + \frac{7}{24}}{1 - \frac{2}{11} \cdot \frac{7}{24}} \quad \left[\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \right]$$

$$= \tan^{-1} \frac{48+77}{11 \times 24 - 14}$$

$$= \tan^{-1} \frac{125}{250} = \tan^{-1} \frac{1}{2} = \text{R.H.S.}$$

Question 4:

Prove $2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{31}{17}$

Answer

To prove: $2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{31}{17}$

$$\begin{aligned}
 \text{L.H.S.} &= 2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7} \\
 &= \tan^{-1} \frac{2 \cdot \frac{1}{2}}{1 - \left(\frac{1}{2}\right)^2} + \tan^{-1} \frac{1}{7} \quad \left[2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2} \right] \\
 &= \tan^{-1} \frac{1}{\left(\frac{3}{4}\right)} + \tan^{-1} \frac{1}{7} \\
 &= \tan^{-1} \frac{4}{3} + \tan^{-1} \frac{1}{7} \\
 &= \tan^{-1} \frac{\frac{4}{3} + \frac{1}{7}}{1 - \frac{4}{3} \cdot \frac{1}{7}} \quad \left[\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \right] \\
 &= \tan^{-1} \frac{\left(\frac{28+3}{21}\right)}{\left(\frac{21-4}{21}\right)} \\
 &= \tan^{-1} \frac{31}{17} = \text{R.H.S.}
 \end{aligned}$$

Question 5:

Write the function in the simplest form:

$$\tan^{-1} \frac{\sqrt{1+x^2} - 1}{x}, \quad x \neq 0$$

Answer

$$\begin{aligned} & \tan^{-1} \frac{\sqrt{1+x^2}-1}{x} \\ \text{Put } x &= \tan \theta \Rightarrow \theta = \tan^{-1} x \\ \therefore \tan^{-1} \frac{\sqrt{1+x^2}-1}{x} &= \tan^{-1} \left(\frac{\sqrt{1+\tan^2 \theta}-1}{\tan \theta} \right) \\ &= \tan^{-1} \left(\frac{\sec \theta - 1}{\tan \theta} \right) = \tan^{-1} \left(\frac{1 - \cos \theta}{\sin \theta} \right) \\ &= \tan^{-1} \left(\frac{2 \sin^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \right) \\ &= \tan^{-1} \left(\tan \frac{\theta}{2} \right) = \frac{\theta}{2} = \frac{1}{2} \tan^{-1} x \end{aligned}$$

Question 6:

Write the function in the simplest form:

$$\tan^{-1} \frac{1}{\sqrt{x^2-1}}, \quad |x| > 1$$

Answer

$$\tan^{-1} \frac{1}{\sqrt{x^2-1}}, \quad |x| > 1$$

Put $x = \operatorname{cosec} \theta \Rightarrow \theta = \operatorname{cosec}^{-1} x$

$$\therefore \tan^{-1} \frac{1}{\sqrt{x^2-1}} = \tan^{-1} \frac{1}{\sqrt{\operatorname{cosec}^2 \theta - 1}}$$

$$= \tan^{-1} \left(\frac{1}{\cot \theta} \right) = \tan^{-1} (\tan \theta)$$

$$= \theta = \operatorname{cosec}^{-1} x = \frac{\pi}{2} - \sec^{-1} x \quad \left[\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2} \right]$$

Question 7:

Write the function in the simplest form:

$$\tan^{-1} \left(\sqrt{\frac{1-\cos x}{1+\cos x}} \right), x < \pi$$

Answer

$$\tan^{-1} \left(\sqrt{\frac{1-\cos x}{1+\cos x}} \right), x < \pi$$

$$\tan^{-1} \left(\sqrt{\frac{1-\cos x}{1+\cos x}} \right) = \tan^{-1} \left(\sqrt{\frac{2 \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}}} \right)$$

$$= \tan^{-1} \left(\frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \right) = \tan^{-1} \left(\tan \frac{x}{2} \right)$$

$$= \frac{x}{2}$$

Question 8:

Write the function in the simplest form:

$$\tan^{-1} \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right), 0 < x < \pi$$

Answer

$$\tan^{-1} \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right)$$

$$= \tan^{-1} \left(\frac{1 - \frac{\sin x}{\cos x}}{1 + \frac{\sin x}{\cos x}} \right)$$

$$= \tan^{-1} \left(\frac{1 - \tan x}{1 + \tan x} \right)$$

$$= \tan^{-1}(1) - \tan^{-1}(\tan x)$$

$$\left[\tan^{-1} \frac{x-y}{1-xy} = \tan^{-1} x - \tan^{-1} y \right]$$

$$= \frac{\pi}{4} - x$$

Question 9:

Write the function in the simplest form:

$$\tan^{-1} \frac{x}{\sqrt{a^2 - x^2}}, \quad |x| < a$$

Answer

$$\tan^{-1} \frac{x}{\sqrt{a^2 - x^2}}$$

$$\text{Put } x = a \sin \theta \Rightarrow \frac{x}{a} = \sin \theta \Rightarrow \theta = \sin^{-1} \left(\frac{x}{a} \right)$$

$$\therefore \tan^{-1} \frac{x}{\sqrt{a^2 - x^2}} = \tan^{-1} \left(\frac{a \sin \theta}{\sqrt{a^2 - a^2 \sin^2 \theta}} \right)$$

$$= \tan^{-1} \left(\frac{a \sin \theta}{a \sqrt{1 - \sin^2 \theta}} \right) = \tan^{-1} \left(\frac{a \sin \theta}{a \cos \theta} \right)$$

$$= \tan^{-1} (\tan \theta) = \theta = \sin^{-1} \frac{x}{a}$$

Question 10:

Write the function in the simplest form:

$$\tan^{-1} \left(\frac{3a^2x - x^3}{a^3 - 3ax^2} \right), \quad a > 0; \quad \frac{-a}{\sqrt{3}} \leq x \leq \frac{a}{\sqrt{3}}$$

Answer

$$\tan^{-1} \left(\frac{3a^2x - x^3}{a^3 - 3ax^2} \right)$$

$$\text{Put } x = a \tan \theta \Rightarrow \frac{x}{a} = \tan \theta \Rightarrow \theta = \tan^{-1} \frac{x}{a}$$

$$\tan^{-1} \left(\frac{3a^2x - x^3}{a^3 - 3ax^2} \right) = \tan^{-1} \left(\frac{3a^2 \cdot a \tan \theta - a^3 \tan^3 \theta}{a^3 - 3a \cdot a^2 \tan^2 \theta} \right)$$

$$= \tan^{-1} \left(\frac{3a^3 \tan \theta - a^3 \tan^3 \theta}{a^3 - 3a^3 \tan^2 \theta} \right)$$

$$= \tan^{-1} \left(\frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} \right)$$

$$= \tan^{-1} (\tan 3\theta)$$

$$= 3\theta$$

$$= 3 \tan^{-1} \frac{x}{a}$$

Question 11:

$$\tan^{-1} \left[2 \cos \left(2 \sin^{-1} \frac{1}{2} \right) \right]$$

Find the value of

Answer

$$\text{Let } \sin^{-1} \frac{1}{2} = x. \text{ Then, } \sin x = \frac{1}{2} = \sin \left(\frac{\pi}{6} \right).$$

$$\therefore \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$$

$$\therefore \tan^{-1} \left[2 \cos \left(2 \sin^{-1} \frac{1}{2} \right) \right] = \tan^{-1} \left[2 \cos \left(2 \times \frac{\pi}{6} \right) \right]$$

$$= \tan^{-1} \left[2 \cos \frac{\pi}{3} \right] = \tan^{-1} \left[2 \times \frac{1}{2} \right]$$

$$= \tan^{-1} 1 = \frac{\pi}{4}$$

Question 12:

Find the value of $\cot(\tan^{-1} a + \cot^{-1} a)$

Answer

$$\begin{aligned} & \cot(\tan^{-1} a + \cot^{-1} a) \\ &= \cot\left(\frac{\pi}{2}\right) \quad \left[\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\right] \\ &= 0 \end{aligned}$$

Question 13:

Find the value of $\tan \frac{1}{2} \left[\sin^{-1} \frac{2x}{1+x^2} + \cos^{-1} \frac{1-y^2}{1+y^2} \right]$, $|x| < 1$, $y > 0$ and $xy < 1$

Answer

Let $x = \tan \theta$. Then, $\theta = \tan^{-1} x$.

$$\therefore \sin^{-1} \frac{2x}{1+x^2} = \sin^{-1} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right) = \sin^{-1} (\sin 2\theta) = 2\theta = 2 \tan^{-1} x$$

Let $y = \tan \phi$. Then, $\phi = \tan^{-1} y$.

$$\therefore \cos^{-1} \frac{1-y^2}{1+y^2} = \cos^{-1} \left(\frac{1 - \tan^2 \phi}{1 + \tan^2 \phi} \right) = \cos^{-1} (\cos 2\phi) = 2\phi = 2 \tan^{-1} y$$

$$\therefore \tan \frac{1}{2} \left[\sin^{-1} \frac{2x}{1+x^2} + \cos^{-1} \frac{1-y^2}{1+y^2} \right]$$

$$= \tan \frac{1}{2} [2 \tan^{-1} x + 2 \tan^{-1} y]$$

$$= \tan [\tan^{-1} x + \tan^{-1} y]$$

$$= \tan \left[\tan^{-1} \left(\frac{x+y}{1-xy} \right) \right]$$

$$= \frac{x+y}{1-xy}$$

Question 14:

If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$, then find the value of x .

Answer

$$\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$$

$$\Rightarrow \sin\left(\sin^{-1}\frac{1}{5}\right)\cos\left(\cos^{-1}x\right) + \cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right) = 1$$

$$[\sin(A+B) = \sin A \cos B + \cos A \sin B]$$

$$\Rightarrow \frac{1}{5} \times x + \cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right) = 1$$

$$\Rightarrow \frac{x}{5} + \cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right) = 1 \quad \dots(1)$$

Now, let $\sin^{-1}\frac{1}{5} = y$.

$$\text{Then, } \sin y = \frac{1}{5} \Rightarrow \cos y = \sqrt{1 - \left(\frac{1}{5}\right)^2} = \frac{2\sqrt{6}}{5} \Rightarrow y = \cos^{-1}\left(\frac{2\sqrt{6}}{5}\right).$$

$$\therefore \sin^{-1}\frac{1}{5} = \cos^{-1}\left(\frac{2\sqrt{6}}{5}\right) \quad \dots(2)$$

Let $\cos^{-1}x = z$.

$$\text{Then, } \cos z = x \Rightarrow \sin z = \sqrt{1 - x^2} \Rightarrow z = \sin^{-1}\left(\sqrt{1 - x^2}\right).$$

$$\therefore \cos^{-1}x = \sin^{-1}\left(\sqrt{1 - x^2}\right) \quad \dots(3)$$

From (1), (2), and (3) we have:

$$\frac{x}{5} + \cos\left(\cos^{-1}\frac{2\sqrt{6}}{5}\right) \cdot \sin\left(\sin^{-1}\sqrt{1 - x^2}\right) = 1$$

$$\Rightarrow \frac{x}{5} + \frac{2\sqrt{6}}{5} \cdot \sqrt{1 - x^2} = 1$$

$$\Rightarrow x + 2\sqrt{6}\sqrt{1 - x^2} = 5$$

$$\Rightarrow 2\sqrt{6}\sqrt{1 - x^2} = 5 - x$$

On squaring both sides, we get:

$$(4)(6)(1-x^2) = 25 + x^2 - 10x$$

$$\Rightarrow 24 - 24x^2 = 25 + x^2 - 10x$$

$$\Rightarrow 25x^2 - 10x + 1 = 0$$

$$\Rightarrow (5x-1)^2 = 0$$

$$\Rightarrow (5x-1) = 0$$

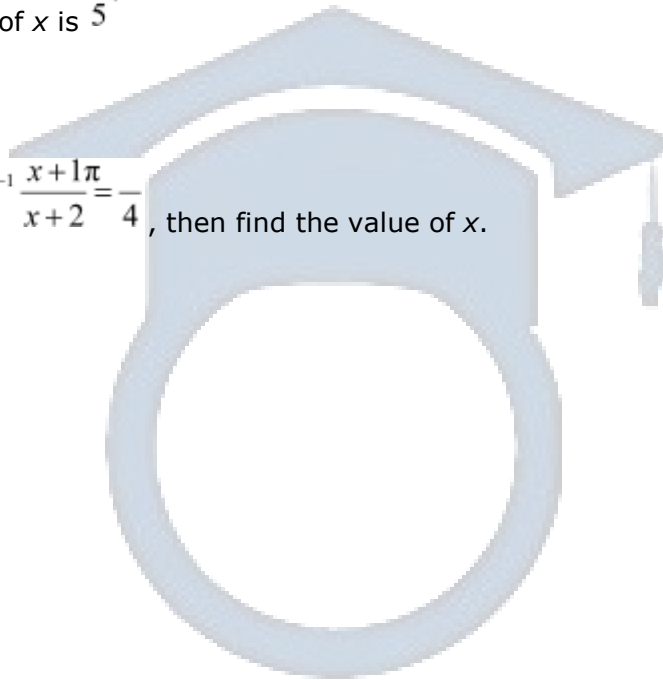
$$\Rightarrow x = \frac{1}{5}$$

Hence, the value of x is $\frac{1}{5}$.

Question 15:

If $\tan^{-1} \frac{x-1}{x-2} + \tan^{-1} \frac{x+1}{x+2} = \frac{\pi}{4}$, then find the value of x .

Answer



$$\tan^{-1} \frac{x-1}{x-2} + \tan^{-1} \frac{x+1}{x+2} = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \left[\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)} \right] = \frac{\pi}{4} \quad \left[\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \right]$$

$$\Rightarrow \tan^{-1} \left[\frac{(x-1)(x+2) + (x+1)(x-2)}{(x+2)(x-2) - (x-1)(x+1)} \right] = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \left[\frac{x^2 + x - 2 + x^2 - x - 2}{x^2 - 4 - x^2 + 1} \right] = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \left[\frac{2x^2 - 4}{-3} \right] = \frac{\pi}{4}$$

$$\Rightarrow \tan \left[\tan^{-1} \frac{4-2x^2}{3} \right] = \tan \frac{\pi}{4}$$

$$\Rightarrow \frac{4-2x^2}{3} = 1$$

$$\Rightarrow 4-2x^2 = 3$$

$$\Rightarrow 2x^2 = 4-3 = 1$$

$$\Rightarrow x = \pm \frac{1}{\sqrt{2}}$$

Hence, the value of x is

$$\pm \frac{1}{\sqrt{2}}$$

Question 16:

Find the values of $\sin^{-1} \left(\sin \frac{2\pi}{3} \right)$

Answer

$$\sin^{-1} \left(\sin \frac{2\pi}{3} \right)$$

We know that $\sin^{-1}(\sin x) = x$ if $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is the principal value branch of $\sin^{-1}x$.

Here, $\frac{2\pi}{3} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Now, $\sin^{-1}\left(\sin \frac{2\pi}{3}\right)$ can be written as:

$$\sin^{-1}\left(\sin \frac{2\pi}{3}\right) = \sin^{-1}\left[\sin\left(\pi - \frac{2\pi}{3}\right)\right] = \sin^{-1}\left(\sin \frac{\pi}{3}\right) \text{ where } \frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\therefore \sin^{-1}\left(\sin \frac{2\pi}{3}\right) = \sin^{-1}\left(\sin \frac{\pi}{3}\right) = \frac{\pi}{3}$$

Question 17:

Find the values of $\tan^{-1}\left(\tan \frac{3\pi}{4}\right)$

Answer

$$\tan^{-1}\left(\tan \frac{3\pi}{4}\right)$$

We know that $\tan^{-1}(\tan x) = x$ if $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which is the principal value branch of $\tan^{-1}x$.

Here, $\frac{3\pi}{4} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Now, $\tan^{-1}\left(\tan \frac{3\pi}{4}\right)$ can be written as:

$$\begin{aligned} \tan^{-1}\left(\tan \frac{3\pi}{4}\right) &= \tan^{-1}\left[-\tan\left(\frac{-3\pi}{4}\right)\right] = \tan^{-1}\left[-\tan\left(\pi - \frac{\pi}{4}\right)\right] \\ &= \tan^{-1}\left[-\tan \frac{\pi}{4}\right] = \tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right] \text{ where } -\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{aligned}$$

$$\therefore \tan^{-1}\left(\tan \frac{3\pi}{4}\right) = \tan^{-1}\left[\tan\left(\frac{-\pi}{4}\right)\right] = \frac{-\pi}{4}$$

Question 18:

Find the values of $\tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$

Answer

Let $\sin^{-1}\frac{3}{5} = x$. Then, $\sin x = \frac{3}{5} \Rightarrow \cos x = \sqrt{1 - \sin^2 x} = \frac{4}{5} \Rightarrow \sec x = \frac{5}{4}$.

$$\therefore \tan x = \sqrt{\sec^2 x - 1} = \sqrt{\frac{25}{16} - 1} = \frac{3}{4}$$

$$\therefore x = \tan^{-1}\frac{3}{4}$$

$$\therefore \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4} \quad \dots(i)$$

$$\text{Now, } \cot^{-1}\frac{3}{2} = \tan^{-1}\frac{2}{3} \quad \dots(ii)$$

$$\left[\tan^{-1}\frac{1}{x} = \cot^{-1}x\right]$$

$$\text{Hence, } \tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$$

$$= \tan\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right)$$

[Using (i) and (ii)]

$$= \tan\left(\tan^{-1}\frac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{3}{4} \cdot \frac{2}{3}}\right)$$

$$\left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan\left(\tan^{-1}\frac{9+8}{12-6}\right)$$

$$= \tan\left(\tan^{-1}\frac{17}{6}\right) = \frac{17}{6}$$

Question 19:

Find the values of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ is equal to

- (A) $\frac{7\pi}{6}$ (B) $\frac{5\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$

Answer

We know that $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$, which is the principal value branch of $\cos^{-1}x$.

Here, $\frac{7\pi}{6} \notin x \in [0, \pi]$.

Now, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ can be written as:

$$\begin{aligned}\cos^{-1}\left(\cos\frac{7\pi}{6}\right) &= \cos^{-1}\left(\cos\frac{-7\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi - \frac{7\pi}{6}\right)\right] \quad [\cos(2\pi + x) = \cos x] \\ &= \cos^{-1}\left[\cos\frac{5\pi}{6}\right] \quad \text{where } \frac{5\pi}{6} \in [0, \pi]\end{aligned}$$

$$\therefore \cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\frac{5\pi}{6}\right) = \frac{5\pi}{6}$$

The correct answer is B.

Question 20:

Find the values of $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right)$ is equal to

- (A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) 1

Answer

Let $\sin^{-1}\left(\frac{-1}{2}\right) = x$. Then, $\sin x = \frac{-1}{2} = -\sin\frac{\pi}{6} = \sin\left(\frac{-\pi}{6}\right)$.

We know that the range of the principal value branch of \sin^{-1} is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

$$\therefore \sin^{-1}\left(\frac{-1}{2}\right) = \frac{-\pi}{6}$$

$$\therefore \sin\left(\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right) = \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\left(\frac{3\pi}{6}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

The correct answer is D.

