Exercise 6.3

Question 1:

Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 4$.

Answer

The given curve is $y = 3x^4 - 4x$.

Then, the slope of the tangent to the given curve at $x = 4$ is given by,

$$\left.\frac{dy}{dx}\right]_{x=4} = 12x^3 - 4\Big]_{x=4} = 12(4)^3 - 4 = 12(64) - 4 = 764$$

Question 2:

Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$.

Answer

The given curve is $y = \frac{x-1}{x-2}$.

$$\therefore \frac{dy}{dx} = \frac{(x-2)(1)-(x-1)(1)}{(x-2)^2}$$

$$= \frac{x-2-x+1}{(x-2)^2} = \frac{-1}{(x-2)^2}$$

Thus, the slope of the tangent at $x = 10$ is given by,

$$\left.\frac{dy}{dx}\right]_{x=10} = \left.\frac{-1}{(x-2)^2}\right]_{x=10} = \frac{-1}{(10-2)^2} = \frac{-1}{64}$$

Hence, the slope of the tangent at $x = 10$ is $\frac{-1}{64}$.

Question 3:

Find the slope of the tangent to curve $y = x^3 - x + 1$ at the point whose x-coordinate is 2.

Answer

The given curve is $y = x^3 - x + 1$.

$$\therefore \frac{dy}{dx} = 3x^2 - 1$$

The slope of the tangent to a curve at (x_0, y_0) is $\left.\frac{dy}{dx}\right]_{(x_0, y_0)}$.

It is given that $x_0 = 2$.

Hence, the slope of the tangent at the point where the x-coordinate is 2 is given by,

$$\left.\frac{dy}{dx}\right]_{x=2} = 3x^2 - 1\Big]_{x=2} = 3(2)^2 - 1 = 12 - 1 = 11$$

Question 4:

Find the slope of the tangent to the curve $y = x^3 - 3x + 2$ at the point whose x-coordinate is 3.

Answer

The given curve is $y = x^3 - 3x + 2$.

$$\therefore \frac{dy}{dx} = 3x^2 - 3$$

The slope of the tangent to a curve at (x_0, y_0) is $\left.\frac{dy}{dx}\right]_{(x_0, y_0)}$.

Hence, the slope of the tangent at the point where the x-coordinate is 3 is given by,

$$\left.\frac{dy}{dx}\right]_{x=3} = 3x^2 - 3\Big]_{x=3} = 3(3)^2 - 3 = 27 - 3 = 24$$

Question 5:

Find the slope of the normal to the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ at $\theta = \frac{\pi}{4}$.

Answer

It is given that $x = a\cos^3\theta$ and $y = a\sin^3\theta$.

$$\therefore \frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta) = -3a\cos^2\theta\sin\theta$$

$$\frac{dy}{d\theta} = 3a\sin^2\theta(\cos\theta)$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

Therefore, the slope of the tangent at $\theta = \frac{\pi}{4}$ is given by,

$$\left.\frac{dy}{dx}\right]_{\theta=\frac{\pi}{4}} = -\tan\theta\Big]_{\theta=\frac{\pi}{4}} = -\tan\frac{\pi}{4} = -1$$

Hence, the slope of the normal at $\theta = \frac{\pi}{4}$ is given by,

$$\frac{1}{\text{slope of the tangent at } \theta = \frac{\pi}{4}} = \frac{-1}{-1} = 1$$

Question 6:

Find the slope of the normal to the curve $x = 1 - a\sin\theta$, $y = b\cos^2\theta$ at $\theta = \frac{\pi}{2}$.

Answer

It is given that $x = 1 - a\sin\theta$ and $y = b\cos^2\theta$.

$$\therefore \frac{dx}{d\theta} = -a\cos\theta \text{ and } \frac{dy}{d\theta} = 2b\cos\theta(-\sin\theta) = -2b\sin\theta\cos\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-2b\sin\theta\cos\theta}{-a\cos\theta} = \frac{2b}{a}\sin\theta$$

Therefore, the slope of the tangent at $\theta = \frac{\pi}{2}$ is given by,

$$\left.\frac{dy}{dx}\right]_{\theta=\frac{\pi}{2}} = \left.\frac{2b}{a}\sin\theta\right]_{\theta=\frac{\pi}{2}} = \frac{2b}{a}\sin\frac{\pi}{2} = \frac{2b}{a}$$

Hence, the slope of the normal at $\theta = \frac{\pi}{2}$ is given by,

$$\frac{1}{\text{slope of the tangent at } \theta = \frac{\pi}{4}} = \frac{-1}{\left(\frac{2b}{a}\right)} = -\frac{a}{2b}$$

Question 7:

Find points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the x-axis.

Answer

The equation of the given curve is $y = x^3 - 3x^2 - 9x + 7.$

$$\therefore \frac{dy}{dx} = 3x^2 - 6x - 9$$

Now, the tangent is parallel to the x-axis if the slope of the tangent is zero.

$$\therefore 3x^2 - 6x - 9 = 0 \Rightarrow x^2 - 2x - 3 = 0$$
$$\Rightarrow (x-3)(x+1) = 0$$
$$\Rightarrow x = 3 \text{ or } x = -1$$

When $x = 3$, $y = (3)^3 - 3(3)^2 - 9(3) + 7 = 27 - 27 - 27 + 7 = -20$.

When $x = -1$, $y = (-1)^3 - 3(-1)^2 - 9(-1) + 7 = -1 - 3 + 9 + 7 = 12$.

Hence, the points at which the tangent is parallel to the x-axis are (3, −20) and (−1, 12).

Question 8:

Find a point on the curve $y = (x - 2)^2$ at which the tangent is parallel to the chord joining the points (2, 0) and (4, 4).

Answer

If a tangent is parallel to the chord joining the points (2, 0) and (4, 4), then the slope of the tangent = the slope of the chord.

The slope of the chord is $\frac{4-0}{4-2}=\frac{4}{2}=2.$

Now, the slope of the tangent to the given curve at a point (x, y) is given by,

$$\frac{dy}{dx}=2(x-2)$$

Since the slope of the tangent = slope of the chord, we have:

$$2(x-2)=2$$

$$\Rightarrow x-2=1 \Rightarrow x=3$$

When $x=3,\ y=(3-2)^2=1.$

Hence, the required point is (3, 1).

Question 9:

Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent is $y = x - 11$.

Answer

The equation of the given curve is $y = x^3 - 11x + 5$.

The equation of the tangent to the given curve is given as $y = x - 11$ (which is of the form $y = mx + c$).

∴Slope of the tangent = 1

Now, the slope of the tangent to the given curve at the point (x, y) is given by,

$$\frac{dy}{dx}=3x^2-11$$

Then, we have:

$$3x^2-11=1$$

$$\Rightarrow 3x^2=12$$

$$\Rightarrow x^2=4$$

$$\Rightarrow x=\pm 2$$

When $x = 2$, $y = (2)^3 - 11(2) + 5 = 8 - 22 + 5 = -9$.

When $x = -2$, $y = (-2)^3 - 11(-2) + 5 = -8 + 22 + 5 = 19$.

Hence, the required points are (2, −9) and (−2, 19).

Question 10:

Find the equation of all lines having slope −1 that are tangents to the curve

$y=\frac{1}{x-1},\ x\neq 1$.

Answer

The equation of the given curve is $y=\frac{1}{x-1},\ x\neq 1$.

The slope of the tangents to the given curve at any point (*x*, *y*) is given by,

$$\frac{dy}{dx}=\frac{-1}{(x-1)^2}$$

If the slope of the tangent is −1, then we have:

$$\frac{-1}{(x-1)^2}=-1$$
$$\Rightarrow (x-1)^2=1$$
$$\Rightarrow x-1=\pm 1$$
$$\Rightarrow x=2,\ 0$$

When *x* = 0, *y* = −1 and when *x* = 2, *y* = 1.

Thus, there are two tangents to the given curve having slope −1. These are passing through the points (0, −1) and (2, 1).

∴The equation of the tangent through (0, −1) is given by,

$$y-(-1)=-1(x-0)$$
$$\Rightarrow y+1=-x$$
$$\Rightarrow y+x+1=0$$

∴The equation of the tangent through (2, 1) is given by,

$y - 1 = -1\ (x - 2)$

$\Rightarrow y - 1 = -x + 2$

$\Rightarrow y + x - 3 = 0$

Hence, the equations of the required lines are $y + x + 1 = 0$ and $y + x - 3 = 0$.

Question 11:

Find the equation of all lines having slope 2 which are tangents to the curve $y = \frac{1}{x-3},\ x \neq 3$.

Answer

The equation of the given curve is $y = \frac{1}{x-3},\ x \neq 3$.

The slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = \frac{-1}{(x-3)^2}$$

If the slope of the tangent is 2, then we have:

$$\frac{-1}{(x-3)^2} = 2$$

$\Rightarrow 2(x-3)^2 = -1$

$\Rightarrow (x-3)^2 = \frac{-1}{2}$

This is not possible since the L.H.S. is positive while the R.H.S. is negative.

Hence, there is no tangent to the given curve having slope 2.

Question 12:

Find the equations of all lines having slope 0 which are tangent to the curve

$y = \frac{1}{x^2 - 2x + 3}$.

Answer

The equation of the given curve is $y = \frac{1}{x^2 - 2x + 3}$.

The slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = \frac{-(2x-2)}{(x^2-2x+3)^2} = \frac{-2(x-1)}{(x^2-2x+3)^2}$$

If the slope of the tangent is 0, then we have:

$$\frac{-2(x-1)}{(x^2-2x+3)^2} = 0$$

$\Rightarrow -2(x-1) = 0$

$\Rightarrow x = 1$

When x = 1, $y = \frac{1}{1-2+3} = \frac{1}{2}$.

∴The equation of the tangent through $\left(1, \frac{1}{2}\right)$ is given by,

$$y-\frac{1}{2}=0(x-1)$$

$$\Rightarrow y-\frac{1}{2}=0$$

$$\Rightarrow y=\frac{1}{2}$$

Hence, the equation of the required line is $y=\frac{1}{2}$.

Question 13:

Find points on the curve $\frac{x^2}{9}+\frac{y^2}{16}=1$ at which the tangents are

(i) parallel to *x*-axis (ii) parallel to *y*-axis

Answer

The equation of the given curve is $\frac{x^2}{9}+\frac{y^2}{16}=1$.

On differentiating both sides with respect to *x*, we have:

$$\frac{2x}{9}+\frac{2y}{16}\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-16x}{9y}$$

(i) The tangent is parallel to the *x*-axis if the slope of the tangent is i.e., 0 $\frac{-16x}{9y}=0,$ which is possible if $x = 0$.

Then, $\frac{x^2}{9}+\frac{y^2}{16}=1$ for $x = 0$

$$\Rightarrow y^2=16 \Rightarrow y=\pm 4$$

Hence, the points at which the tangents are parallel to the *x*-axis are

(0, 4) and (0, − 4).

(ii) The tangent is parallel to the y-axis if the slope of the normal is 0, which

gives $\frac{-1}{\left(\frac{-16x}{9y}\right)} = \frac{9y}{16x} = 0$ $\Rightarrow y = 0$.

Then, $\frac{x^2}{9} + \frac{y^2}{16} = 1$ for $y = 0$.

$\Rightarrow x = \pm 3$

Hence, the points at which the tangents are parallel to the y-axis are

(3, 0) and (− 3, 0).

Question 14:

Find the equations of the tangent and normal to the given curves at the indicated points:

(i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (0, 5)

(ii) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (1, 3)

(iii) $y = x^3$ at (1, 1)

(iv) $y = x^2$ at (0, 0)

(v) $x = \cos t$, $y = \sin t$ at $t = \frac{\pi}{4}$

Answer

(i) The equation of the curve is $y = x^4 - 6x^3 + 13x^2 - 10x + 5$.

On differentiating with respect to x, we get:

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

$$\left.\frac{dy}{dx}\right]_{(0,\ 5)} = -10$$

Thus, the slope of the tangent at (0, 5) is −10. The equation of the tangent is given as:

$y - 5 = -10(x - 0)$

$\Rightarrow y - 5 = -10x$

$\Rightarrow 10x + y = 5$

The slope of the normal at (0, 5) is $\dfrac{-1}{\text{Slope of the tangent at }(0, 5)} = \dfrac{1}{10}$.

Therefore, the equation of the normal at (0, 5) is given as:

$$y - 5 = \frac{1}{10}(x - 0)$$
$$\Rightarrow 10y - 50 = x$$
$$\Rightarrow x - 10y + 50 = 0$$

(ii) The equation of the curve is $y = x^4 - 6x^3 + 13x^2 - 10x + 5$.

On differentiating with respect to x, we get:

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$
$$\left.\frac{dy}{dx}\right]_{(1,\ 3)} = 4 - 18 + 26 - 10 = 2$$

Thus, the slope of the tangent at (1, 3) is 2. The equation of the tangent is given as:

$$y - 3 = 2(x - 1)$$
$$\Rightarrow y - 3 = 2x - 2$$
$$\Rightarrow y = 2x + 1$$

The slope of the normal at (1, 3) is $\dfrac{-1}{\text{Slope of the tangent at }(1, 3)} = \dfrac{-1}{2}$.

Therefore, the equation of the normal at (1, 3) is given as:

$$y-3=-\frac{1}{2}(x-1)$$
$$\Rightarrow 2y-6=-x+1$$
$$\Rightarrow x+2y-7=0$$

(iii) The equation of the curve is $y = x^3$.

On differentiating with respect to x, we get:

$$\frac{dy}{dx}=3x^2$$
$$\left.\frac{dy}{dx}\right]_{(1,\ 1)}=3(1)^2=3$$

Thus, the slope of the tangent at (1, 1) is 3 and the equation of the tangent is given as:

$$y-1=3(x-1)$$
$$\Rightarrow y=3x-2$$

The slope of the normal at (1, 1) is $\dfrac{-1}{\text{Slope of the tangent at } (1,\ 1)}=\dfrac{-1}{3}$.

Therefore, the equation of the normal at (1, 1) is given as:

$$y-1=\frac{-1}{3}(x-1)$$
$$\Rightarrow 3y-3=-x+1$$
$$\Rightarrow x+3y-4=0$$

(iv) The equation of the curve is $y = x^2$.

On differentiating with respect to x, we get:

$$\frac{dy}{dx}=2x$$
$$\left.\frac{dy}{dx}\right]_{(0,\ 0)}=0$$

Thus, the slope of the tangent at (0, 0) is 0 and the equation of the tangent is given as:

$y - 0 = 0\ (x - 0)$

$\Rightarrow y = 0$

The slope of the normal at (0, 0) is $\dfrac{-1}{\text{Slope of the tangent at }(0,0)}=-\dfrac{1}{0}$, which is not defined.

Therefore, the equation of the normal at (x_0, y_0) = (0, 0) is given by

$x = x_0 = 0.$

(v) The equation of the curve is $x = \cos t$, $y = \sin t$.

$$x=\cos t \text{ and } y=\sin t$$

$$\therefore \frac{dx}{dt}=-\sin t,\ \frac{dy}{dt}=\cos t$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\cos t}{-\sin t}=-\cot t$$

$$\left.\frac{dy}{dx}\right]_{t=\frac{\pi}{4}}=-\cot t=-1$$

☐The slope of the tangent at $t=\frac{\pi}{4}$ is −1.

When $t=\frac{\pi}{4},\ x=\frac{1}{\sqrt{2}}$ and $y=\frac{1}{\sqrt{2}}$.

Thus, the equation of the tangent to the given curve at $t=\frac{\pi}{4}$ i.e., at $\left[\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)\right]$ is

$$y-\frac{1}{\sqrt{2}}=-1\left(x-\frac{1}{\sqrt{2}}\right).$$

$$\Rightarrow x+y-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=0$$

$$\Rightarrow x+y-\sqrt{2}=0$$

The slope of the normal at $t=\frac{\pi}{4}$ is $\dfrac{-1}{\text{Slope of the tangent at } t=\frac{\pi}{4}}=1.$

Therefore, the equation of the normal to the given curve at $t=\frac{\pi}{4}$ i.e., at $\left[\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right]$ is

$$y-\frac{1}{\sqrt{2}}=1\left(x-\frac{1}{\sqrt{2}}\right).$$

$$\Rightarrow x=y$$

Question 15:

Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$ which is

(a) parallel to the line $2x - y + 9 = 0$

(b) perpendicular to the line $5y - 15x = 13$.

Answer

The equation of the given curve is $y=x^2-2x+7$.

On differentiating with respect to x, we get:

$$\frac{dy}{dx}=2x-2$$

(a) The equation of the line is $2x - y + 9 = 0$.

$2x - y + 9 = 0$ ☐ $y = 2x + 9$

This is of the form $y = mx + c$.

☐Slope of the line = 2

If a tangent is parallel to the line $2x - y + 9 = 0$, then the slope of the tangent is equal to the slope of the line.

Therefore, we have:

$2 = 2x - 2$

$$\Rightarrow 2x=4$$

$$\Rightarrow x=2$$

Now, $x = 2$

$\Rightarrow y = 4 - 4 + 7 = 7$

Thus, the equation of the tangent passing through (2, 7) is given by,

$$y-7=2(x-2)$$

$$\Rightarrow y-2x-3=0$$

Hence, the equation of the tangent line to the given curve (which is parallel to line 2*x* − *y* + 9 = 0) is $y-2x-3=0$.

(b) The equation of the line is 5*y* − 15*x* = 13.

5*y* − 15*x* = 13 ☐ $y=3x+\frac{13}{5}$

This is of the form *y* = *mx* + *c*.

☐Slope of the line = 3

If a tangent is perpendicular to the line 5*y* − 15*x* = 13, then the slope of the tangent is

$$\frac{-1}{\text{slope of the line}}=\frac{-1}{3}.$$

$$\Rightarrow 2x-2=\frac{-1}{3}$$

$$\Rightarrow 2x=\frac{-1}{3}+2$$

$$\Rightarrow 2x=\frac{5}{3}$$

$$\Rightarrow x=\frac{5}{6}$$

$$\text{Now, } x=\frac{5}{6}$$

$$\Rightarrow y=\frac{25}{36}-\frac{10}{6}+7=\frac{25-60+252}{36}=\frac{217}{36}$$

Thus, the equation of the tangent passing through $\left(\frac{5}{6},\ \frac{217}{36}\right)$ is given by,

$$y-\frac{217}{36}=-\frac{1}{3}\left(x-\frac{5}{6}\right)$$

$$\Rightarrow \frac{36y-217}{36}=\frac{-1}{18}(6x-5)$$

$$\Rightarrow 36y-217=-2(6x-5)$$

$$\Rightarrow 36y-217=-12x+10$$

$$\Rightarrow 36y+12x-227=0$$

Hence, the equation of the tangent line to the given curve (which is perpendicular to line $5y - 15x = 13$) is $36y + 12x - 227 = 0$.

Question 16:

Show that the tangents to the curve $y = 7x^3 + 11$ at the points where $x = 2$ and $x = -2$ are parallel.

Answer

The equation of the given curve is $y = 7x^3 + 11$.

$$\therefore \frac{dy}{dx} = 21x^2$$

The slope of the tangent to a curve at (x_0, y_0) is $\left.\frac{dy}{dx}\right]_{(x_0, y_0)}$.

Therefore, the slope of the tangent at the point where $x = 2$ is given by,

$$\left.\frac{dy}{dx}\right]_{x=-2} = 21(2)^2 = 84$$

It is observed that the slopes of the tangents at the points where $x = 2$ and $x = -2$ are equal.

Hence, the two tangents are parallel.

Question 17:

Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.

Answer

The equation of the given curve is $y = x^3$.

$$\therefore \frac{dy}{dx} = 3x^2$$

The slope of the tangent at the point (x, y) is given by,

$$\left.\frac{dy}{dx}\right]_{(x,y)} = 3x^2$$

When the slope of the tangent is equal to the y-coordinate of the point, then $y = 3x^2$.

Also, we have $y = x^3$.

□ $3x^2 = x^3$

□ $x^2(x - 3) = 0$

□ $x = 0, x = 3$

When $x = 0$, then $y = 0$ and when $x = 3$, then $y = 3(3)^2 = 27$.

Hence, the required points are (0, 0) and (3, 27).

Question 18: For the curve $y = 4x^3 - 2x^5$, find all the points at which the tangents passes through the origin.

Answer

The equation of the given curve is $y = 4x^3 - 2x^5$.

$$\therefore \frac{dy}{dx} = 12x^2 - 10x^4$$

Therefore, the slope of the tangent at a point (x, y) is $12x^2 - 10x^4$.

The equation of the tangent at (x, y) is given by,

$$Y - y = \left(12x^2 - 10x^4\right)(X - x) \quad \ldots(1)$$

When the tangent passes through the origin (0, 0), then $X = Y = 0$.

Therefore, equation (1) reduces to:

$$-y = \left(12x^2 - 10x^4\right)(-x)$$

$$y = 12x^3 - 10x^5$$

Also, we have

$$y = 4x^3 - 2x^5.$$

$$\therefore 12x^3 - 10x^5 = 4x^3 - 2x^5$$

$$\Rightarrow 8x^5 - 8x^3 = 0$$

$$\Rightarrow x^5 - x^3 = 0$$

$$\Rightarrow x^3\left(x^2 - 1\right) = 0$$

$$\Rightarrow x = 0, \ \pm 1$$

When $x = 0$, $y = 4(0)^3 - 2(0)^5 = 0$.

When $x = 1$, $y = 4(1)^3 - 2(1)^5 = 2$.

When $x = -1$, $y = 4(-1)^3 - 2(-1)^5 = -2$.

Hence, the required points are (0, 0), (1, 2), and (−1, −2).

Question 19:

Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to the x-axis.

Answer

The equation of the given curve is $x^2 + y^2 - 2x - 3 = 0$.

On differentiating with respect to x, we have:

$$2x + 2y\frac{dy}{dx} - 2 = 0$$

$$\Rightarrow y\frac{dy}{dx} = 1 - x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-x}{y}$$

Now, the tangents are parallel to the x-axis if the slope of the tangent is 0.

$$\therefore \frac{1-x}{y} = 0 \Rightarrow 1 - x = 0 \Rightarrow x = 1$$

But, $x^2 + y^2 - 2x - 3 = 0$ for $x = 1$.

$\Rightarrow y^2 = 4$ □ $y = \pm 2$

Hence, the points at which the tangents are parallel to the x-axis are (1, 2) and (1, −2).

Question 20:

Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.

Answer

The equation of the given curve is $ay^2 = x^3$.

On differentiating with respect to x, we have:

$$2ay\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{2ay}$$

The slope of a tangent to the curve at (x_0, y_0) is $\left.\frac{dy}{dx}\right]_{(x_0, y_0)}$.

$\Rightarrow$ The slope of the tangent to the given curve at (am^2, am^3) is

$$\left.\frac{dy}{dx}\right]_{(am^2,\ am^3)} = \frac{3\left(am^2\right)^2}{2a\left(am^3\right)} = \frac{3a^2m^4}{2a^2m^3} = \frac{3m}{2}.$$

☐ Slope of normal at (am^2, am^3)

$$= \frac{-1}{\text{slope of the tangent at }\left(am^2,\ am^3\right)} = \frac{-2}{3m}$$

Hence, the equation of the normal at (am^2, am^3) is given by,

$$y - am^3 = \frac{-2}{3m}\left(x - am^2\right)$$

$$\Rightarrow 3my - 3am^4 = -2x + 2am^2$$

$$\Rightarrow 2x + 3my - am^2\left(2 + 3m^2\right) = 0$$

Question 21:

Find the equation of the normals to the curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$.

Answer

The equation of the given curve is $y = x^3 + 2x + 6$.

The slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = 3x^2 + 2$$

☐ Slope of the normal to the given curve at any point (x, y)

$$= \frac{-1}{\text{Slope of the tangent at the point }(x,\ y)}$$

$$= \frac{-1}{3x^2 + 2}$$

The equation of the given line is $x + 14y + 4 = 0$.

$x + 14y + 4 = 0$ □ $y = -\frac{1}{14}x - \frac{4}{14}$ (which is of the form $y = mx + c$)

□Slope of the given line = $\frac{-1}{14}$

If the normal is parallel to the line, then we must have the slope of the normal being equal to the slope of the line.

$\therefore \frac{-1}{3x^2+2} = \frac{-1}{14}$

$\Rightarrow 3x^2 + 2 = 14$

$\Rightarrow 3x^2 = 12$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

When $x = 2$, $y = 8 + 4 + 6 = 18$.

When $x = -2$, $y = -8 - 4 + 6 = -6$.

Therefore, there are two normals to the given curve with slope $\frac{-1}{14}$ and passing through the points (2, 18) and (−2, −6).

Thus, the equation of the normal through (2, 18) is given by,

$y - 18 = \frac{-1}{14}(x-2)$

$\Rightarrow 14y - 252 = -x + 2$

$\Rightarrow x + 14y - 254 = 0$

And, the equation of the normal through (−2, −6) is given by,

$y - (-6) = \frac{-1}{14}\left[x - (-2)\right]$

$\Rightarrow y + 6 = \frac{-1}{14}(x+2)$

$\Rightarrow 14y + 84 = -x - 2$

$\Rightarrow x + 14y + 86 = 0$

Hence, the equations of the normals to the given curve (which are parallel to the given line) are $x + 14y - 254 = 0$ and $x + 14y + 86 = 0$.

Question 22:

Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

Answer

The equation of the given parabola is $y^2 = 4ax$.

On differentiating $y^2 = 4ax$ with respect to x, we have:

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

□The slope of the tangent at $(at^2,\ 2at)$ is $\left.\frac{dy}{dx}\right]_{(at^2,\ 2at)} = \frac{2a}{2at} = \frac{1}{t}$.

Then, the equation of the tangent at $(at^2,\ 2at)$ is given by,

$$y - 2at = \frac{1}{t}\left(x - at^2\right)$$

$$\Rightarrow ty - 2at^2 = x - at^2$$

$$\Rightarrow ty = x + at^2$$

Now, the slope of the normal at $(at^2,\ 2at)$ is given by,

$$\frac{-1}{\text{Slope of the tangent at } \left(at^2, 2at\right)} = -t$$

Thus, the equation of the normal at $(at^2, 2at)$ is given as:

$$y - 2at = -t\left(x - at^2\right)$$

$$\Rightarrow y - 2at = -tx + at^3$$

$$\Rightarrow y = -tx + 2at + at^3$$

Question 23:

Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$. [**Hint**: Two curves intersect at right angle if the tangents to the curves at the point of intersection are perpendicular to each other.]

Answer

The equations of the given curves are given as $x = y^2$ and $xy = k$.

Putting $x = y^2$ in $xy = k$, we get:

$$y^3 = k \Rightarrow y = k^{\frac{1}{3}}$$

$$\therefore x = k^{\frac{2}{3}}$$

Thus, the point of intersection of the given curves is $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$.

Differentiating $x = y^2$ with respect to x, we have:

$$1 = 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$$

Therefore, the slope of the tangent to the curve $x = y^2$ at $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$ is $\left.\frac{dy}{dx}\right]_{\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)} = \frac{1}{2k^{\frac{1}{3}}}$.

On differentiating $xy = k$ with respect to x, we have:

$$x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

☐ Slope of the tangent to the curve $xy = k$ at $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$ is given by,

$$\left.\frac{dy}{dx}\right]_{\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)} = \left.\frac{-y}{x}\right]_{\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)} = -\frac{k^{\frac{1}{3}}}{k^{\frac{2}{3}}} = \frac{-1}{k^{\frac{1}{3}}}$$

We know that two curves intersect at right angles if the tangents to the curves at the point of intersection i.e., at $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$ are perpendicular to each other.

This implies that we should have the product of the tangents as − 1.

Thus, the given two curves cut at right angles if the product of the slopes of their respective tangents at $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$ is −1.

i.e., $\left(\frac{1}{2k^{\frac{1}{3}}}\right)\left(\frac{-1}{k^{\frac{1}{3}}}\right)=-1$

$\Rightarrow 2k^{\frac{2}{3}}=1$

$\Rightarrow \left(2k^{\frac{2}{3}}\right)^3=(1)^3$

$\Rightarrow 8k^2=1$

Hence, the given two curves cut at right angels if $8k^2 = 1$.

Question 24:

Find the equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (x_0, y_0).

Answer

Differentiating $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with respect to x, we have:

$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$

$\Rightarrow \frac{2y}{b^2}\frac{dy}{dx}=\frac{2x}{a^2}$

$\Rightarrow \frac{dy}{dx}=\frac{b^2x}{a^2y}$

Therefore, the slope of the tangent at (x_0, y_0) is $\left.\frac{dy}{dx}\right]_{(x_0, y_0)}=\frac{b^2x_0}{a^2y_0}$.

Then, the equation of the tangent at (x_0, y_0) is given by,

$$y-y_0=\frac{b^2x_0}{a^2y_0}(x-x_0)$$

$$\Rightarrow a^2yy_0-a^2y_0^2=b^2xx_0-b^2x_0^2$$

$$\Rightarrow b^2xx_0-a^2yy_0-b^2x_0^2+a^2y_0^2=0$$

$$\Rightarrow \frac{xx_0}{a^2}-\frac{yy_0}{b^2}-\left(\frac{x_0^2}{a^2}-\frac{y_0^2}{b^2}\right)=0 \qquad \left[\text{On dividing both sides by } a^2b^2\right]$$

$$\Rightarrow \frac{xx_0}{a^2}-\frac{yy_0}{b^2}-1=0 \qquad \left[(x_0,y_0) \text{ lies on the hyperbola } \frac{x^2}{a^2}-\frac{y^2}{b^2}=1\right]$$

$$\Rightarrow \frac{xx_0}{a^2}-\frac{yy_0}{b^2}=1$$

Now, the slope of the normal at (x_0, y_0) is given by,

$$\frac{-1}{\text{Slope of the tangent at } (x_0, y_0)}=\frac{-a^2y_0}{b^2x_0}$$

Hence, the equation of the normal at (x_0, y_0) is given by,

$$y-y_0=\frac{-a^2y_0}{b^2x_0}(x-x_0)$$

$$\Rightarrow \frac{y-y_0}{a^2y_0}=\frac{-(x-x_0)}{b^2x_0}$$

$$\Rightarrow \frac{y-y_0}{a^2y_0}+\frac{(x-x_0)}{b^2x_0}=0$$

Question 25:

Find the equation of the tangent to the curve $y=\sqrt{3x-2}$ which is parallel to the line $4x - 2y + 5 = 0$.

Answer

The equation of the given curve is $y=\sqrt{3x-2}.$

The slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx}=\frac{3}{2\sqrt{3x-2}}$$

The equation of the given line is 4x − 2y + 5 = 0.

4x − 2y + 5 = 0 ☐ $y=2x+\frac{5}{2}$ (which is of the form $y=mx+c$)

☐Slope of the line = 2

Now, the tangent to the given curve is parallel to the line 4x − 2y − 5 = 0 if the slope of the tangent is equal to the slope of the line.

$$\frac{3}{2\sqrt{3x-2}}=2$$

$$\Rightarrow \sqrt{3x-2}=\frac{3}{4}$$

$$\Rightarrow 3x-2=\frac{9}{16}$$

$$\Rightarrow 3x=\frac{9}{16}+2=\frac{41}{16}$$

$$\Rightarrow x=\frac{41}{48}$$

When $x=\frac{41}{48},\ y=\sqrt{3\left(\frac{41}{48}\right)-2}=\sqrt{\frac{41}{16}-2}=\sqrt{\frac{41-32}{16}}=\sqrt{\frac{9}{16}}=\frac{3}{4}.$

☐Equation of the tangent passing through the point $\left(\frac{41}{48},\ \frac{3}{4}\right)$ is given by,

$$y-\frac{3}{4}=2\left(x-\frac{41}{48}\right)$$

$$\Rightarrow \frac{4y-3}{4}=2\left(\frac{48x-41}{48}\right)$$

$$\Rightarrow 4y-3=\frac{48x-41}{6}$$

$$\Rightarrow 24y-18=48x-41$$

$$\Rightarrow 48x-24y=23$$

Hence, the equation of the required tangent is $48x-24y=23$.

Question 26:

The slope of the normal to the curve y = 2x^2 + 3 sin x at x = 0 is

(A) 3 (B) $\frac{1}{3}$ (C) −3 (D) $-\frac{1}{3}$

Answer

The equation of the given curve is $y = 2x^2 + 3\sin x$.

Slope of the tangent to the given curve at $x = 0$ is given by,

$$\left.\frac{dy}{dx}\right]_{x=0} = 4x + 3\cos x\Big]_{x=0} = 0 + 3\cos 0 = 3$$

Hence, the slope of the normal to the given curve at $x = 0$ is

$$\frac{-1}{\text{Slope of the tangent at } x = 0} = \frac{-1}{3}.$$

The correct answer is D.

Question 27:

The line $y = x + 1$ is a tangent to the curve $y^2 = 4x$ at the point

(A) (1, 2) (B) (2, 1) (C) (1, −2) (D) (−1, 2)

Answer

The equation of the given curve is $y^2 = 4x$.

Differentiating with respect to x, we have:

$$2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$$

Therefore, the slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = \frac{2}{y}$$

The given line is $y = x + 1$ (which is of the form $y = mx + c$)

☐ Slope of the line = 1

The line $y = x + 1$ is a tangent to the given curve if the slope of the line is equal to the slope of the tangent. Also, the line must intersect the curve.

Thus, we must have:

$$\frac{2}{y}=1$$

$$\Rightarrow y=2$$

Now, $y=x+1 \Rightarrow x=y-1 \Rightarrow x=2-1=1$

Hence, the line $y = x + 1$ is a tangent to the given curve at the point (1, 2).

The correct answer is A.