Question 1:

Find $\left| \vec{a} \times \vec{b} \right|$, if $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$.

Answer

We have,

$\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix}$$

$$= \hat{i}(-14+14) - \hat{j}(2-21) + \hat{k}(-2+21) = 19\hat{j} + 19\hat{k}$$

$$\therefore \left| \vec{a} \times \vec{b} \right| = \sqrt{(19)^2 + (19)^2} = \sqrt{2 \times (19)^2} = 19\sqrt{2}$$

Question 2:

Find a unit vector perpendicular to each of the vector $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, where
$\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$.

Answer

We have,

$\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$

$\therefore \vec{a} + \vec{b} = 4\hat{i} + 4\hat{j},\ \vec{a} - \vec{b} = 2\hat{i} + 4\hat{k}$

$$(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix} = \hat{i}(16) - \hat{j}(16) + \hat{k}(-8) = 16\hat{i} - 16\hat{j} - 8\hat{k}$$

$$\therefore \left| (\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) \right| = \sqrt{16^2 + (-16)^2 + (-8)^2}$$

$$= \sqrt{2^2 \times 8^2 + 2^2 \times 8^2 + 8^2}$$

$$= 8\sqrt{2^2 + 2^2 + 1} = 8\sqrt{9} = 8 \times 3 = 24$$

Hence, the unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ is given by the relation,

$$= \pm \frac{\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)}{\left|\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)\right|} = \pm\frac{16\hat{i}-16\hat{j}-8\hat{k}}{24}$$

$$= \pm\frac{2\hat{i}-2\hat{j}-\hat{k}}{3} = \pm\frac{2}{3}\hat{i}\mp\frac{2}{3}\hat{j}\mp\frac{1}{3}\hat{k}$$

Question 3:

If a unit vector $\vec{a}$ makes an angles $\dfrac{\pi}{3}$ with $\hat{i},\dfrac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the compounds of $\vec{a}$.

Answer

Let unit vector $\vec{a}$ have (a_1, a_2, a_3) components.

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

Since $\vec{a}$ is a unit vector, $\left|\vec{a}\right|=1$.

Also, it is given that $\vec{a}$ makes angles $\dfrac{\pi}{3}$ with $\hat{i},\dfrac{\pi}{4}$ with $\hat{j}$, and an acute angle θ with $\hat{k}$. Then, we have:

$$\cos\frac{\pi}{3} = \frac{a_1}{\left|\vec{a}\right|}$$

$$\Rightarrow \frac{1}{2} = a_1 \qquad \left[\left|\vec{a}\right|=1\right]$$

$$\cos\frac{\pi}{4} = \frac{a_2}{\left|\vec{a}\right|}$$

$$\Rightarrow \frac{1}{\sqrt{2}} = a_2 \qquad \left[\left|\vec{a}\right|=1\right]$$

Also, $\cos\theta = \dfrac{a_3}{\left|\vec{a}\right|}$.

$$\Rightarrow a_3 = \cos\theta$$

Now,

$|a| = 1$

$\Rightarrow \sqrt{a_1^2 + a_2^2 + a_3^2} = 1$

$\Rightarrow \left(\dfrac{1}{2}\right)^2 + \left(\dfrac{1}{\sqrt{2}}\right)^2 + \cos^2 \theta = 1$

$\Rightarrow \dfrac{1}{4} + \dfrac{1}{2} + \cos^2 \theta = 1$

$\Rightarrow \dfrac{3}{4} + \cos^2 \theta = 1$

$\Rightarrow \cos^2 \theta = 1 - \dfrac{3}{4} = \dfrac{1}{4}$

$\Rightarrow \cos \theta = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{3}$

$\therefore a_3 = \cos\dfrac{\pi}{3} = \dfrac{1}{2}$

Hence, $\theta = \dfrac{\pi}{3}$ and the components of $\vec{a}$ are $\left(\dfrac{1}{2}, \dfrac{1}{\sqrt{2}}, \dfrac{1}{2}\right)$.

Question 4:

Show that

$$\left(\vec{a} - \vec{b}\right) \times \left(\vec{a} + \vec{b}\right) = 2\left(\vec{a} \times \vec{b}\right)$$

Answer

$\left(\vec{a} - \vec{b}\right) \times \left(\vec{a} + \vec{b}\right)$

$= \left(\vec{a} - \vec{b}\right) \times \vec{a} + \left(\vec{a} - \vec{b}\right) \times \vec{b}$ [By distributivity of vector product over addition]

$= \vec{a} \times \vec{a} - \vec{b} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{b}$ [Again, by distributivity of vector product over addition]

$= \vec{0} + \vec{a} \times \vec{b} + \vec{a} \times \vec{b} - \vec{0}$

$= 2\vec{a} \times \vec{b}$

Question 5:

Find λ and μ if $\left(2\hat{i} + 6\hat{j} + 27\hat{k}\right) \times \left(\hat{i} + \lambda\hat{j} + \mu\hat{k}\right) = \vec{0}$.

Answer

$$\left(2\hat{i}+6\hat{j}+27\hat{k}\right)\times\left(\hat{i}+\lambda\hat{j}+\mu\hat{k}\right)=\vec{0}$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix}=0\hat{i}+0\hat{j}+0\hat{k}$$

$$\Rightarrow \hat{i}\left(6\mu-27\lambda\right)-\hat{j}\left(2\mu-27\right)+\hat{k}\left(2\lambda-6\right)=0\hat{i}+0\hat{j}+0\hat{k}$$

On comparing the corresponding components, we have:

$$6\mu-27\lambda=0$$

$$2\mu-27=0$$

$$2\lambda-6=0$$

Now,

$$2\lambda-6=0 \Rightarrow \lambda=3$$

$$2\mu-27=0 \Rightarrow \mu=\frac{27}{2}$$

Hence, $\lambda=3$ and $\mu=\frac{27}{2}$.

Question 6:

Given that $\vec{a}\cdot\vec{b}=0$ and $\vec{a}\times\vec{b}=\vec{0}$. What can you conclude about the vectors $\vec{a}$ and $\vec{b}$?

Answer

$$\vec{a}\cdot\vec{b}=0$$

Then,

(i) Either $\left|\vec{a}\right|=0$ or $\left|\vec{b}\right|=0$, or $\vec{a}\perp\vec{b}$ (in case $\vec{a}$ and $\vec{b}$ are non-zero)

$$\vec{a}\times\vec{b}=0$$

(ii) Either $\left|\vec{a}\right|=0$ or $\left|\vec{b}\right|=0$, or $\vec{a}\parallel\vec{b}$ (in case $\vec{a}$ and $\vec{b}$ are non-zero)

But, $\vec{a}$ and $\vec{b}$ cannot be perpendicular and parallel simultaneously.

Hence, $\left|\vec{a}\right|=0$ or $\left|\vec{b}\right|=0$.

Question 7:

Let the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ given as $a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, $c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$. Then show

that $=\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

Answer

We have,

$\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k},\ \vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$

$\left(\vec{b}+\vec{c}\right)=\left(b_1+c_1\right)\hat{i}+\left(b_2+c_2\right)\hat{j}+\left(b_3+c_3\right)\hat{k}$

Now, $\vec{a}\times\left(\vec{b}+\vec{c}\right)\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \end{vmatrix}$

$=\hat{i}\left[a_2\left(b_3+c_3\right)-a_3\left(b_2+c_2\right)\right]-\hat{j}\left[a_1\left(b_3+c_3\right)-a_3\left(b_1+c_1\right)\right]+\hat{k}\left[a_1\left(b_2+c_2\right)-a_2\left(b_1+c_1\right)\right]$

$=\hat{i}\left[a_2b_3+a_2c_3-a_3b_2-a_3c_2\right]+\hat{j}\left[-a_1b_3-a_1c_3+a_3b_1+a_3c_1\right]+\hat{k}\left[a_1b_2+a_1c_2-a_2b_1-a_2c_1\right]$ ...(1)

$\vec{a}\times\vec{b}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$=\hat{i}\left[a_2b_3-a_3b_2\right]+\hat{j}\left[b_1a_3-a_1b_3\right]+\hat{k}\left[a_1b_2-a_2b_1\right]$ (2)

$\vec{a}\times\vec{c}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$=\hat{i}\left[a_2c_3-a_3c_2\right]+\hat{j}\left[a_3c_1-a_1c_3\right]+\hat{k}\left[a_1c_2-a_2c_1\right]$ (3)

On adding (2) and (3), we get:

$\left(\vec{a}\times\vec{b}\right)+\left(\vec{a}\times\vec{c}\right)=\hat{i}\left[a_2b_3+a_2c_3-a_3b_2-a_3c_2\right]+\hat{j}\left[b_1a_3+a_3c_1-a_1b_3-a_1c_3\right]$

$+\hat{k}\left[a_1b_2+a_1c_2-a_2b_1-a_2c_1\right]$ (4)

Now, from (1) and (4), we have:

$\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

Hence, the given result is proved.

Question 8:

If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a} \times \vec{b} = \vec{0}$. Is the converse true? Justify your answer with an example.

Answer

Take any parallel non-zero vectors so that $\vec{a} \times \vec{b} = \vec{0}$.

Let $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}, \ \vec{b} = 4\hat{i} + 6\hat{j} + 8\hat{k}.$

Then,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 4 & 6 & 8 \end{vmatrix} = \hat{i}(24-24) - \hat{j}(16-16) + \hat{k}(12-12) = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

It can now be observed that:

$$|\vec{a}| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$$

$$\therefore \vec{a} \neq \vec{0}$$

$$|\vec{b}| = \sqrt{4^2 + 6^2 + 8^2} = \sqrt{116}$$

$$\therefore \vec{b} \neq \vec{0}$$

Hence, the converse of the given statement need not be true.

Question 9:

Find the area of the triangle with vertices A (1, 1, 2), B (2, 3, 5) and C (1, 5, 5).

Answer

The vertices of triangle ABC are given as A (1, 1, 2), B (2, 3, 5), and C (1, 5, 5).

The adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of ΔABC are given as:

$$\overrightarrow{AB} = (2-1)\hat{i} + (3-1)\hat{j} + (5-2)\hat{k} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\overrightarrow{BC} = (1-2)\hat{i} + (5-3)\hat{j} + (5-5)\hat{k} = -\hat{i} + 2\hat{j}$$

Area of $\triangle ABC = \dfrac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 2 & 0 \end{vmatrix} = \hat{i}(-6) - \hat{j}(3) + \hat{k}(2+2) = -6\hat{i} - 3\hat{j} + 4\hat{k}$$

$$\therefore \left|\overrightarrow{AB} \times \overrightarrow{BC}\right| = \sqrt{(-6)^2 + (-3)^2 + 4^2} = \sqrt{36+9+16} = \sqrt{61}$$

Hence, the area of $\triangle ABC$ is $\dfrac{\sqrt{61}}{2}$ square units.

Question 10:

Find the area of the parallelogram whose adjacent sides are determined by the vector

$\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.

Answer

The area of the parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $\left|\vec{a} \times \vec{b}\right|$.

Adjacent sides are given as:

$\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$

$$\therefore \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix} = \hat{i}(-1+21) - \hat{j}(1-6) + \hat{k}(-7+2) = 20\hat{i} + 5\hat{j} - 5\hat{k}$$

$$\left|\vec{a} \times \vec{b}\right| = \sqrt{20^2 + 5^2 + 5^2} = \sqrt{400+25+25} = 15\sqrt{2}$$

Hence, the area of the given parallelogram is $15\sqrt{2}$ square units.

Question 11:

Let the vectors $\vec{a}$ and $\vec{b}$ be such that $\left|\vec{a}\right| = 3$ and $\left|\vec{b}\right| = \dfrac{\sqrt{2}}{3}$, then $\vec{a} \times \vec{b}$ is a unit vector, if

the angle between $\vec{a}$ and $\vec{b}$ is

(A) $\dfrac{\pi}{6}$ (B) $\dfrac{\pi}{4}$ (C) $\dfrac{\pi}{3}$ (D) $\dfrac{\pi}{2}$

Answer

It is given that $\left|\vec{a}\right| = 3$ and $\left|\vec{b}\right| = \dfrac{\sqrt{2}}{3}$.

We know that $\vec{a} \times \vec{b} = \left|\vec{a}\right|\left|\vec{b}\right| \sin\theta \, \hat{n}$, where $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and θ is the angle between $\vec{a}$ and $\vec{b}$.

Now, $\vec{a} \times \vec{b}$ is a unit vector if $\left|\vec{a} \times \vec{b}\right| = 1$.

$\left|\vec{a} \times \vec{b}\right| = 1$

$\Rightarrow \left|\left|\vec{a}\right|\left|\vec{b}\right| \sin\theta \, \hat{n}\right| = 1$

$\Rightarrow \left|\vec{a}\right|\left|\vec{b}\right|\left|\sin\theta\right| = 1$

$\Rightarrow 3 \times \dfrac{\sqrt{2}}{3} \times \sin\theta = 1$

$\Rightarrow \sin\theta = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \theta = \dfrac{\pi}{4}$

Hence, $\vec{a} \times \vec{b}$ is a unit vector if the angle between $\vec{a}$ and $\vec{b}$ is $\dfrac{\pi}{4}$.

The correct answer is B.

Question 12:

Area of a rectangle having vertices A, B, C, and D with position vectors

$-\hat{i} + \dfrac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} + \dfrac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} - \dfrac{1}{2}\hat{j} + 4\hat{k}$ and $-\hat{i} - \dfrac{1}{2}\hat{j} + 4\hat{k}$ respectively is

(A) $\dfrac{1}{2}$ (B) 1

(C) 2 (D) 4

Answer

The position vectors of vertices A, B, C, and D of rectangle ABCD are given as:

$$\overrightarrow{OA} = -\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k},\ \overrightarrow{OB} = \hat{i} + \frac{1}{2}\hat{j} + 4\hat{k},\ \overrightarrow{OC} = \hat{i} - \frac{1}{2}\hat{j} + 4\hat{k},\ \overrightarrow{OD} = -\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$$

The adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of the given rectangle are given as:

$$\overrightarrow{AB} = (1+1)\hat{i} + \left(\frac{1}{2} - \frac{1}{2}\right)\hat{j} + (4-4)\hat{k} = 2\hat{i}$$

$$\overrightarrow{BC} = (1-1)\hat{i} + \left(-\frac{1}{2} - \frac{1}{2}\right)\hat{j} + (4-4)\hat{k} = -\hat{j}$$

$$\therefore \overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix} = \hat{k}(-2) = -2\hat{k}$$

$$\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{(-2)^2} = 2$$

Now, it is known that the area of a parallelogram whose adjacent sides are

$\vec{a}$ and $\vec{b}$ is $\left|\vec{a} \times \vec{b}\right|$.

Hence, the area of the given rectangle is $\left|\overrightarrow{AB} \times \overrightarrow{BC}\right| = 2$ square units.

The correct answer is C.