Exercise 7.10

Question 1:

$$\int_0^1 \frac{x}{x^2+1}dx$$

Answer

$$\int_0^1 \frac{x}{x^2+1}dx$$

Let $x^2+1=t \Rightarrow 2x\,dx=dt$

When x = 0, t = 1 and when x = 1, t = 2

$$\therefore \int_0^1 \frac{x}{x^2+1}dx = \frac{1}{2}\int_1^2 \frac{dt}{t}$$

$$= \frac{1}{2}\left[\log|t|\right]_1^2$$

$$= \frac{1}{2}\left[\log 2 - \log 1\right]$$

$$= \frac{1}{2}\log 2$$

Question 2:

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\, d\phi$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\, d\phi = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^4\phi\cos\phi\, d\phi$$

Also, let $\sin\phi = t \Rightarrow \cos\phi\, d\phi = dt$

When $\phi = 0$, $t = 0$ and when $\phi = \frac{\pi}{2}$, $t = 1$

$$\therefore I = \int_0^1 \sqrt{t}\left(1-t^2\right)^2 dt$$

$$= \int_0^1 t^{\frac{1}{2}}\left(1+t^4-2t^2\right)dt$$

$$= \int_0^1 \left[t^{\frac{1}{2}} + t^{\frac{9}{2}} - 2t^{\frac{5}{2}}\right]dt$$

$$= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{11}{2}}}{\frac{11}{2}} - \frac{2t^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1$$

$$= \frac{2}{3} + \frac{2}{11} - \frac{4}{7}$$

$$= \frac{154+42-132}{231}$$

$$= \frac{64}{231}$$

Question 3:

$$\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$$

Answer

$$\text{Let } I = \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$$

Also, let $x = \tan\theta$ □ $dx = \sec^2\theta\, d\theta$

When $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \frac{\pi}{4}$

$$I = \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sin^{-1}(\sin 2\theta)\sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\theta\cdot\sec^2\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \theta\cdot\sec^2\theta\, d\theta$$

Taking θ as first function and $\sec^2\theta$ as second function and integrating by parts, we obtain

$$I = 2\left[\theta\int\sec^2\theta\, d\theta - \int\left\{\left(\frac{d}{dx}\theta\right)\int\sec^2\theta\, d\theta\right\}d\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\theta\tan\theta - \int\tan\theta\, d\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\theta\tan\theta + \log|\cos\theta|\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\frac{\pi}{4}\tan\frac{\pi}{4} + \log\left|\cos\frac{\pi}{4}\right| - \log|\cos 0|\right]$$

$$= 2\left[\frac{\pi}{4} + \log\left(\frac{1}{\sqrt{2}}\right) - \log 1\right]$$

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2\right]$$

$$= \frac{\pi}{2} - \log 2$$

Question 4:

$$\int_0^2 x\sqrt{x+2} \quad \left(\text{Put } x+2=t^2\right)$$

Answer

$$\int_0^2 x\sqrt{x+2}dx$$

Let $x + 2 = t^2$ □ $dx = 2tdt$

When $x = 0$, $t=\sqrt{2}$ and when $x = 2$, $t = 2$

$$\therefore \int_0^2 x\sqrt{x+2}\,dx = \int_{\sqrt{2}}^2 \left(t^2-2\right)\sqrt{t^2}\,2t\,dt$$
$$= 2\int_{\sqrt{2}}^2 \left(t^2-2\right)t^2\,dt$$
$$= 2\int_{\sqrt{2}}^2 \left(t^4-2t^2\right)dt$$
$$= 2\left[\frac{t^5}{5}-\frac{2t^3}{3}\right]_{\sqrt{2}}^2$$
$$= 2\left[\frac{32}{5}-\frac{16}{3}-\frac{4\sqrt{2}}{5}+\frac{4\sqrt{2}}{3}\right]$$
$$= 2\left[\frac{96-80-12\sqrt{2}+20\sqrt{2}}{15}\right]$$
$$= 2\left[\frac{16+8\sqrt{2}}{15}\right]$$
$$= \frac{16\left(2+\sqrt{2}\right)}{15}$$
$$= \frac{16\sqrt{2}\left(\sqrt{2}+1\right)}{15}$$

Question 5:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx$$

Answer

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx$$

Let cos x = t □ −sin$x$ dx = dt

When x = 0, t = 1 and when $x=\frac{\pi}{2},\ t=0$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx = -\int_1^0 \frac{dt}{1+t^2}$$

$$= -\left[\tan^{-1} t\right]_1^0$$

$$= -\left[\tan^{-1} 0 - \tan^{-1} 1\right]$$

$$= -\left[-\frac{\pi}{4}\right]$$

$$= \frac{\pi}{4}$$

Question 6:

$$\int_0^2 \frac{dx}{x+4-x^2}$$

Answer

$$\int_0^2 \frac{dx}{x+4-x^2} = \int_0^2 \frac{dx}{-\left(x^2-x-4\right)}$$

$$= \int_0^2 \frac{dx}{-\left(x^2-x+\frac{1}{4}-\frac{1}{4}-4\right)}$$

$$= \int_0^2 \frac{dx}{-\left[\left(x-\frac{1}{2}\right)^2-\frac{17}{4}\right]}$$

$$= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}$$

Let $x-\frac{1}{2}=t$ ☐ $dx = dt$

When $x=0,\ t=-\frac{1}{2}$ and when $x=2,\ t=\frac{3}{2}$

$$\therefore \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}=\int_{-\frac{1}{2}}^{\frac{3}{2}}\frac{dt}{\left(\frac{\sqrt{17}}{2}\right)^2-t^2}$$

$$=\left[\frac{1}{2\left(\frac{\sqrt{17}}{2}\right)}\log\frac{\frac{\sqrt{17}}{2}+t}{\frac{\sqrt{17}}{2}-t}\right]_{-\frac{1}{2}}^{\frac{3}{2}}$$

$$=\frac{1}{\sqrt{17}}\left[\log\frac{\frac{\sqrt{17}}{2}+\frac{3}{2}}{\frac{\sqrt{17}}{2}-\frac{3}{2}}-\frac{\log\frac{\sqrt{17}}{2}-\frac{1}{2}}{\log\frac{\sqrt{17}}{2}+\frac{1}{2}}\right]$$

$$=\frac{1}{\sqrt{17}}\left[\log\frac{\sqrt{17}+3}{\sqrt{17}-3}-\log\frac{\sqrt{17}-1}{\sqrt{17}+1}\right]$$

$$=\frac{1}{\sqrt{17}}\log\frac{\sqrt{17}+3}{\sqrt{17}-3}\times\frac{\sqrt{17}+1}{\sqrt{17}-1}$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{17+3+4\sqrt{17}}{17+3-4\sqrt{17}}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{20+4\sqrt{17}}{20-4\sqrt{17}}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left(\frac{5+\sqrt{17}}{5-\sqrt{17}}\right)$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{\left(5+\sqrt{17}\right)\left(5+\sqrt{17}\right)}{25-17}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{25+17+10\sqrt{17}}{8}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left(\frac{42+10\sqrt{17}}{8}\right)$$

$$=\frac{1}{\sqrt{17}}\log\left(\frac{21+5\sqrt{17}}{4}\right)$$

Question 7:

$$\int_{-1}^{1} \frac{dx}{x^2+2x+5}$$

Answer

$$\int_{-1}^{1} \frac{dx}{x^2+2x+5} = \int_{-1}^{1} \frac{dx}{\left(x^2+2x+1\right)+4} = \int_{-1}^{1} \frac{dx}{\left(x+1\right)^2+\left(2\right)^2}$$

Let $x + 1 = t$ □ $dx = dt$

When $x = -1$, $t = 0$ and when $x = 1$, $t = 2$

$$\therefore \int_{-1}^{1} \frac{dx}{\left(x+1\right)^2+\left(2\right)^2} = \int_{0}^{2} \frac{dt}{t^2+2^2}$$

$$= \left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]_0^2$$

$$= \frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}0$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}$$

Question 8:

$$\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx$$

Answer

$$\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx$$

Let $2x = t$ □ $2dx = dt$

When $x = 1$, $t = 2$ and when $x = 2$, $t = 4$

$$\therefore \int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx = \frac{1}{2} \int_2^4 \left(\frac{2}{t} - \frac{2}{t^2}\right) e^t dt$$

$$= \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2}\right) e^t dt$$

Let $\frac{1}{t} = f(t)$

Then, $f'(t) = -\frac{1}{t^2}$

$$\Rightarrow \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2}\right) e^t dt = \int_2^4 e^t \left[f(t) + f'(t) \right] dt$$

$$= \left[e^t f(t) \right]_2^4$$

$$= \left[e^t \cdot \frac{2}{t} \right]_2^4$$

$$= \left[\frac{e^t}{t} \right]_2^4$$

$$= \frac{e^4}{4} - \frac{e^2}{2}$$

$$= \frac{e^2\left(e^2 - 2\right)}{4}$$

Question 9:

The value of the integral $\int_{\frac{1}{3}}^{1} \frac{\left(x - x^3\right)^{\frac{1}{3}}}{x^4} dx$ is

A. 6

B. 0

C. 3

D. 4

Answer

Let $I = \int_{\frac{1}{3}}^{1} \frac{\left(x - x^3\right)^{\frac{1}{3}}}{x^4} dx$

Also, let $x = \sin\theta \Rightarrow dx = \cos\theta \, d\theta$

When $x = \frac{1}{3}$, $\theta = \sin^{-1}\left(\frac{1}{3}\right)$ and when $x = 1$, $\theta = \frac{\pi}{2}$

$$\Rightarrow I = \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{\left(\sin\theta - \sin^3\theta\right)^{\frac{1}{3}}}{\sin^4\theta} \cos\theta \, d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\sin\theta)^{\frac{1}{3}}\left(1 - \sin^2\theta\right)^{\frac{1}{3}}}{\sin^4\theta} \cos\theta \, d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\sin\theta)^{\frac{1}{3}}(\cos\theta)^{\frac{2}{3}}}{\sin^4\theta} \cos\theta \, d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\sin\theta)^{\frac{1}{3}}(\cos\theta)^{\frac{2}{3}}}{\sin^2\theta \sin^2\theta} \cos\theta \, d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\cos\theta)^{\frac{5}{3}}}{(\sin\theta)^{\frac{5}{3}}} \operatorname{cosec}^2\theta \, d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} (\cot\theta)^{\frac{5}{3}} \operatorname{cosec}^2\theta \, d\theta$$

Let $\cot\theta = t$ □ $-\text{cosec}2\theta \, d\theta = dt$

When $\theta = \sin^{-1}\left(\frac{1}{3}\right)$, $t = 2\sqrt{2}$ and when $\theta = \frac{\pi}{2}$, $t = 0$

$$\therefore I = -\int_{2\sqrt{2}}^{0} (t)^{\frac{5}{3}} dt$$
$$= -\left[\frac{3}{8}(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$
$$= -\frac{3}{8}\left[(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$
$$= -\frac{3}{8}\left[-\left(2\sqrt{2}\right)^{\frac{8}{3}}\right]$$
$$= \frac{3}{8}\left[\left(\sqrt{8}\right)^{\frac{8}{3}}\right]$$
$$= \frac{3}{8}\left[(8)^{\frac{4}{3}}\right]$$
$$= \frac{3}{8}[16]$$
$$= 3 \times 2$$
$$= 6$$

Hence, the correct Answer is A.

Question 10:

If $f(x) = \int_0^x t \sin t \, dt$, then $f'(x)$ is

A. cos *x* + *x* sin *x*

B. *x* sin *x*

C. *x* cos *x*

D. sin *x* + *x* cos *x*

Answer

$$f(x) = \int_0^x t \sin t dt$$

Integrating by parts, we obtain

$$f(x) = t\int_0^x \sin t\,dt - \int_0^x \left\{\left(\frac{d}{dt}t\right)\int \sin t\,dt\right\}dt$$
$$= \left[t(-\cos t)\right]_0^x - \int_0^x (-\cos t)\,dt$$
$$= \left[-t\cos t + \sin t\right]_0^x$$
$$= -x\cos x + \sin x$$

$$\Rightarrow f'(x) = -\left[\left\{x(-\sin x)\right\} + \cos x\right] + \cos x$$
$$= x\sin x - \cos x + \cos x$$
$$= x\sin x$$

Hence, the correct Answer is B.