Exercise 7.4

Question 1:

$$\frac{3x^2}{x^6+1}$$

Answer

Let $x^3 = t$

$\therefore 3x^2\ dx = dt$

$$\Rightarrow \int \frac{3x^2}{x^6+1}dx = \int \frac{dt}{t^2+1}$$
$$= \tan^{1} t + \text{C}$$
$$= \tan^{-1}\left(x^3\right) + \text{C}$$

Question 2:

$$\frac{1}{\sqrt{1+4x^2}}$$

Answer

Let $2x = t$

$\therefore 2dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{1+4x^2}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{1+t^2}}$$

$$= \frac{1}{2}\left[\log\left|t+\sqrt{t^2+1}\right|\right]+C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|\right]$$

$$= \frac{1}{2}\log\left|2x+\sqrt{4x^2+1}\right|+C$$

Question 3:

$$\frac{1}{\sqrt{(2-x)^2+1}}$$

Answer

Let $2 - x = t$

$\Rightarrow -dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{(2-x)^2+1}}dx = -\int \frac{1}{\sqrt{t^2+1}}dt$$

$$= -\log\left|t+\sqrt{t^2+1}\right|+C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|\right]$$

$$= -\log\left|2-x+\sqrt{(2-x)^2+1}\right|+C$$

$$= \log\left|\frac{1}{(2-x)+\sqrt{x^2-4x+5}}\right|+C$$

Question 4:

$$\frac{1}{\sqrt{9-25x^2}}$$

Answer

Let $5x = t$

$\therefore 5dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{9-25x^2}}\,dx = \frac{1}{5}\int \frac{1}{9-t^2}\,dt$$
$$= \frac{1}{5}\int \frac{1}{\sqrt{3^2-t^2}}\,dt$$
$$= \frac{1}{5}\sin^{-1}\left(\frac{t}{3}\right) + C$$
$$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$$

Question 5:

$$\frac{3x}{1+2x^4}$$

Answer

Let $\sqrt{2}x^2 = t$

$\therefore\ 2\sqrt{2}x\ dx = dt$

$$\Rightarrow \int \frac{3x}{1+2x^4}dx = \frac{3}{2\sqrt{2}}\int \frac{dt}{1+t^2}$$
$$= \frac{3}{2\sqrt{2}}\left[\tan^{-1} t\right] + C$$
$$= \frac{3}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}x^2\right) + C$$

Question 6:

$$\frac{x^2}{1-x^6}$$

Answer

Let $x^3 = t$

$\therefore 3x^2\ dx = dt$

$$\Rightarrow \int \frac{x^2}{1-x^6}dx = \frac{1}{3}\int \frac{dt}{1-t^2}$$
$$= \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|\right] + C$$
$$= \frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + C$$

Question 7:

$$\frac{x-1}{\sqrt{x^2-1}}$$

Answer

$$\int \frac{x-1}{\sqrt{x^2-1}}\ dx = \int \frac{x}{\sqrt{x^2-1}}\ dx - \int \frac{1}{\sqrt{x^2-1}}\ dx \qquad \ldots(1)$$

For $\int \frac{x}{\sqrt{x^2-1}}dx$, let $x^2-1=t \Rightarrow 2x\ dx = dt$

$$\therefore \int \frac{x}{\sqrt{x^2-1}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$
$$= \frac{1}{2}\int t^{-\frac{1}{2}}dt$$
$$= \frac{1}{2}\left[2t^{\frac{1}{2}}\right]$$
$$= \sqrt{t}$$
$$= \sqrt{x^2-1}$$

From (1), we obtain

$$\int \frac{x-1}{\sqrt{x^2-1}}dx = \int \frac{x}{\sqrt{x^2-1}}dx - \int \frac{1}{\sqrt{x^2-1}}dx \qquad \left[\int \frac{1}{\sqrt{x^2-a^2}}dt = \log\left|x+\sqrt{x^2-a^2}\right|\right]$$

$$= \sqrt{x^2-1} - \log\left|x+\sqrt{x^2-1}\right| + C$$

Question 8:

$$\frac{x^2}{\sqrt{x^6+a^6}}$$

Answer

Let $x^3 = t$

$\Rightarrow 3x^2\, dx = dt$

$$\therefore \int \frac{x^2}{\sqrt{x^6+a^6}}dx = \frac{1}{3}\int \frac{dt}{\sqrt{t^2+\left(a^3\right)^2}}$$

$$= \frac{1}{3}\log\left|t+\sqrt{t^2+a^6}\right| + C$$

$$= \frac{1}{3}\log\left|x^3+\sqrt{x^6+a^6}\right| + C$$

Question 9:

$$\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}$$

Answer

Let $\tan x = t$

$\therefore \sec^2 x\, dx = dt$

$$\Rightarrow \int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}} dx = \int \frac{dt}{\sqrt{t^2 + 2^2}}$$

$$= \log\left|t + \sqrt{t^2 + 4}\right| + C$$

$$= \log\left|\tan x + \sqrt{\tan^2 x + 4}\right| + C$$

Question 10:

$$\frac{1}{\sqrt{x^2 + 2x + 2}}$$

Answer

$$\int \frac{1}{\sqrt{x^2 + 2x + 2}} dx = \int \frac{1}{\sqrt{(x+1)^2 + (1)^2}} dx$$

Let $x + 1 = t$

$\therefore\ dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{x^2 + 2x + 2}} dx = \int \frac{1}{\sqrt{t^2 + 1}} dt$$

$$= \log\left|t + \sqrt{t^2 + 1}\right| + C$$

$$= \log\left|(x+1) + \sqrt{(x+1)^2 + 1}\right| + C$$

$$= \log\left|(x+1) + \sqrt{x^2 + 2x + 2}\right| + C$$

Question 11:

$$\frac{1}{\sqrt{9x^2 + 6x + 5}}$$

Answer

$$\int \frac{1}{9x^2+6x+5}dx = \int \frac{1}{(3x+1)^2+(2)^2}dx$$

Let $(3x+1)=t$

$\therefore\ 3dx = dt$

$$\Rightarrow \int \frac{1}{(3x+1)^2+(2)^2}dx = \frac{1}{3}\int \frac{1}{t^2+2^2}dt$$

$$= \frac{1}{3}\left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right]+C$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right)+C$$

Question 12:

$$\frac{1}{\sqrt{7-6x-x^2}}$$

Answer

$7-6x-x^2$ can be written as $7-\left(x^2+6x+9-9\right)$.

Therefore,

$7-\left(x^2+6x+9-9\right)$

$=16-\left(x^2+6x+9\right)$

$=16-(x+3)^2$

$=(4)^2-(x+3)^2$

$$\therefore\ \int \frac{1}{\sqrt{7-6x-x^2}}dx = \int \frac{1}{\sqrt{(4)^2-(x+3)^2}}dx$$

Let $x+3=t$

$\Rightarrow\ dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{(4)^2-(x+3)^2}}dx = \int \frac{1}{\sqrt{(4)^2-(t)^2}}dt$$

$$= \sin^{-1}\left(\frac{t}{4}\right)+C$$

$$= \sin^{-1}\left(\frac{x+3}{4}\right)+C$$

Question 13:

$$\frac{1}{\sqrt{(x-1)(x-2)}}$$

Answer

$(x-1)(x-2)$ can be written as x^2-3x+2.

Therefore,

$$x^2-3x+2$$
$$=x^2-3x+\frac{9}{4}-\frac{9}{4}+2$$
$$=\left(x-\frac{3}{2}\right)^2-\frac{1}{4}$$
$$=\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2$$

$$\therefore \int\frac{1}{\sqrt{(x-1)(x-2)}}dx=\int\frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$$

Let $x-\frac{3}{2}=t$

$\therefore\ dx=dt$

$$\Rightarrow \int\frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx=\int\frac{1}{\sqrt{t^2-\left(\frac{1}{2}\right)^2}}dt$$

$$=\log\left|t+\sqrt{t^2-\left(\frac{1}{2}\right)^2}\right|+\text{C}$$

$$=\log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+\text{C}$$

Question 14:

$$\frac{1}{\sqrt{8+3x-x^2}}$$

Answer

$8+3x-x^2$ can be written as $8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$.

Therefore,

$$8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$$

$$=\frac{41}{4}-\left(x-\frac{3}{2}\right)^2$$

$$\Rightarrow \int \frac{1}{\sqrt{8+3x-x^2}}\,dx=\int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}\,dx$$

Let $x-\frac{3}{2}=t$

$\therefore\ dx=dt$

$$\Rightarrow \int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}\,dx=\int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-t^2}}\,dt$$

$$=\sin^{-1}\left(\frac{t}{\frac{\sqrt{41}}{2}}\right)+C$$

$$=\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{41}}{2}}\right)+C$$

$$=\sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C$$

Question 15:

$$\frac{1}{\sqrt{(x-a)(x-b)}}$$

Answer

$(x-a)(x-b)$ can be written as $x^2-(a+b)x+ab$.

Therefore,

$x^2-(a+b)x+ab$

$=x^2-(a+b)x+\frac{(a+b)^2}{4}-\frac{(a+b)^2}{4}+ab$

$=\left[x-\left(\frac{a+b}{2}\right)\right]^2-\frac{(a-b)^2}{4}$

$\Rightarrow\int\frac{1}{\sqrt{(x-a)(x-b)}}dx=\int\frac{1}{\sqrt{\left\{x-\left(\frac{a+b}{2}\right)\right\}^2-\left(\frac{a-b}{2}\right)^2}}dx$

Let $x-\left(\frac{a+b}{2}\right)=t$

$\therefore\ dx=dt$

$\Rightarrow\int\frac{1}{\sqrt{\left\{x-\left(\frac{a+b}{2}\right)\right\}^2-\left(\frac{a-b}{2}\right)^2}}dx=\int\frac{1}{\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}}dt$

$$=\log\left|t+\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}\right|+C$$

$$=\log\left|\left\{x-\left(\frac{a+b}{2}\right)\right\}+\sqrt{(x-a)(x-b)}\right|+C$$

Question 16:

$\frac{4x+1}{\sqrt{2x^2+x-3}}$

Answer

Let $4x+1=A\frac{d}{dx}\left(2x^2+x-3\right)+B$

$\Rightarrow 4x+1=A(4x+1)+B$

$\Rightarrow 4x+1=4Ax+A+B$

Equating the coefficients of *x* and constant term on both sides, we obtain

$4A = 4 \Rightarrow A = 1$

$A + B = 1 \Rightarrow B = 0$

Let $2x^2 + x - 3 = t$

$\therefore (4x + 1)\, dx = dt$

$$\Rightarrow \int \frac{4x+1}{\sqrt{2x^2+x-3}}\,dx = \int \frac{1}{\sqrt{t}}\,dt$$
$$= 2\sqrt{t} + C$$
$$= 2\sqrt{2x^2+x-3} + C$$

Question 17:

$$\frac{x+2}{\sqrt{x^2-1}}$$

Answer

$$\text{Let } x+2 = A\frac{d}{dx}\left(x^2-1\right)+B \qquad \ldots(1)$$
$$\Rightarrow x+2 = A(2x)+B$$

Equating the coefficients of x and constant term on both sides, we obtain

$$2A = 1 \Rightarrow A = \frac{1}{2}$$
$$B = 2$$

From (1), we obtain

$$(x+2)=\frac{1}{2}(2x)+2$$

$$\text{Then, } \int \frac{x+2}{\sqrt{x^2-1}}dx = \int \frac{\frac{1}{2}(2x)+2}{\sqrt{x^2-1}}dx$$

$$= \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx + \int \frac{2}{\sqrt{x^2-1}}dx \qquad ...(2)$$

$$\text{In } \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx, \text{ let } x^2-1=t \Rightarrow 2xdx = dt$$

$$\frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{2}\left[2\sqrt{t}\right]$$

$$= \sqrt{t}$$

$$= \sqrt{x^2-1}$$

$$\text{Then, } \int \frac{2}{\sqrt{x^2-1}}dx = 2\int \frac{1}{\sqrt{x^2-1}}dx = 2\log\left|x+\sqrt{x^2-1}\right|$$

From equation (2), we obtain

$$\int \frac{x+2}{\sqrt{x^2-1}}dx = \sqrt{x^2-1} + 2\log\left|x+\sqrt{x^2-1}\right| + C$$

Question 18:

$$\frac{5x-2}{1+2x+3x^2}$$

Answer

$$\text{Let } 5x-2 = A\frac{d}{dx}\left(1+2x+3x^2\right)+B$$

$$\Rightarrow 5x-2 = A(2+6x)+B$$

Equating the coefficient of *x* and constant term on both sides, we obtain

$$5 = 6A \Rightarrow A = \frac{5}{6}$$

$$2A + B = -2 \Rightarrow B = -\frac{11}{3}$$

$$\therefore 5x - 2 = \frac{5}{6}(2+6x) + \left(-\frac{11}{3}\right)$$

$$\Rightarrow \int \frac{5x-2}{1+2x+3x^2}\,dx = \int \frac{\frac{5}{6}(2+6x) - \frac{11}{3}}{1+2x+3x^2}\,dx$$

$$= \frac{5}{6}\int \frac{2+6x}{1+2x+3x^2}\,dx - \frac{11}{3}\int \frac{1}{1+2x+3x^2}\,dx$$

Let $I_1 = \int \frac{2+6x}{1+2x+3x^2}\,dx$ and $I_2 = \int \frac{1}{1+2x+3x^2}\,dx$

$$\therefore \int \frac{5x-2}{1+2x+3x^2}\,dx = \frac{5}{6}I_1 - \frac{11}{3}I_2 \qquad ...(1)$$

$$I_1 = \int \frac{2+6x}{1+2x+3x^2}\,dx$$

Let $1 + 2x + 3x^2 = t$

$$\Rightarrow (2+6x)\,dx = dt$$

$$\therefore I_1 = \int \frac{dt}{t}$$

$$I_1 = \log|t|$$

$$I_1 = \log\left|1+2x+3x^2\right| \qquad ...(2)$$

$$I_2 = \int \frac{1}{1+2x+3x^2}\,dx$$

$1+2x+3x^2$ can be written as $1+3\left(x^2+\frac{2}{3}x\right)$.

Therefore,

$$1+3\left(x^2+\frac{2}{3}x\right)$$

$$=1+3\left(x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}\right)$$

$$=1+3\left(x+\frac{1}{3}\right)^2-\frac{1}{3}$$

$$=\frac{2}{3}+3\left(x+\frac{1}{3}\right)^2$$

$$=3\left[\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right]$$

$$=3\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]$$

$$I_2=\frac{1}{3}\int\frac{1}{\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]}dx$$

$$=\frac{1}{3}\left[\frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right)\right]$$

$$=\frac{1}{3}\left[\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) \qquad ...(3)$$

Substituting equations (2) and (3) in equation (1), we obtain

$$\int \frac{5x-2}{1+2x+3x^2}dx = \frac{5}{6}\left[\log\left|1+2x+3x^2\right|\right] - \frac{11}{3}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right] + C$$

$$= \frac{5}{6}\log\left|1+2x+3x^2\right| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$$

Question 19:

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}}$$

Answer

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}} = \frac{6x+7}{\sqrt{x^2-9x+20}}$$

Let $6x+7 = A\frac{d}{dx}\left(x^2-9x+20\right)+B$

$\Rightarrow 6x+7 = A(2x-9)+B$

Equating the coefficients of x and constant term, we obtain

$2A = 6 \Rightarrow A = 3$

$-9A + B = 7 \Rightarrow B = 34$

$\therefore 6x + 7 = 3(2x - 9) + 34$

$$\int \frac{6x+7}{\sqrt{x^2-9x+20}} = \int \frac{3(2x-9)+34}{\sqrt{x^2-9x+20}}dx$$

$$= 3\int \frac{2x-9}{\sqrt{x^2-9x+20}}dx + 34\int \frac{1}{\sqrt{x^2-9x+20}}dx$$

Let $I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}}dx$ and $I_2 = \int \frac{1}{\sqrt{x^2-9x+20}}dx$

$$\therefore \int \frac{6x+7}{\sqrt{x^2-9x+20}} = 3I_1 + 34I_2 \qquad ...(1)$$

Then,

$$I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}}dx$$

Let $x^2 - 9x + 20 = t$

$\Rightarrow (2x-9)dx = dt$

$\Rightarrow I_1 = \frac{dt}{\sqrt{t}}$

$I_1 = 2\sqrt{t}$

$$I_1 = 2\sqrt{x^2-9x+20} \qquad ...(2)$$

and $I_2 = \int \frac{1}{\sqrt{x^2-9x+20}}dx$

$x^2-9x+20$ can be written as $x^2-9x+20+\frac{81}{4}-\frac{81}{4}$.

Therefore,

$$x^2-9x+20+\frac{81}{4}-\frac{81}{4}$$

$$=\left(x-\frac{9}{2}\right)^2-\frac{1}{4}$$

$$=\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2$$

$$\Rightarrow I_2=\int\frac{1}{\sqrt{\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$$

$$I_2=\log\left|\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right| \qquad ...(3)$$

Substituting equations (2) and (3) in (1), we obtain

$$\int\frac{6x+7}{\sqrt{x^2-9x+20}}dx=3\left[2\sqrt{x^2-9x+20}\right]+34\log\left[\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right]+C$$

$$=6\sqrt{x^2-9x+20}+34\log\left[\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right]+C$$

Question 20:

$$\frac{x+2}{\sqrt{4x-x^2}}$$

Answer

Let $x+2=A\frac{d}{dx}\left(4x-x^2\right)+B$

$\Rightarrow x+2=A\left(4-2x\right)+B$

Equating the coefficients of *x* and constant term on both sides, we obtain

$-2A=1 \Rightarrow A=-\frac{1}{2}$

$4A+B=2 \Rightarrow B=4$

$\Rightarrow (x+2)=-\frac{1}{2}(4-2x)+4$

$$\therefore \int \frac{x+2}{\sqrt{4x-x^2}}dx = \int \frac{-\frac{1}{2}(4-2x)+4}{\sqrt{4x-x^2}}dx$$

$$= -\frac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}dx + 4\int \frac{1}{\sqrt{4x-x^2}}dx$$

Let $I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx$ and $I_2 \int \frac{1}{\sqrt{4x-x^2}}dx$

$$\therefore \int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}I_1 + 4I_2 \qquad ...(1)$$

Then, $I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx$

Let $4x-x^2=t$

$\Rightarrow (4-2x)dx = dt$

$$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{4x-x^2} \qquad ...(2)$$

$I_2 = \int \frac{1}{\sqrt{4x-x^2}}dx$

$\Rightarrow 4x-x^2 = -(-4x+x^2)$

$= (-4x+x^2+4-4)$

$= 4-(x-2)^2$

$= (2)^2-(x-2)^2$

$$\therefore I_2 = \int \frac{1}{\sqrt{(2)^2-(x-2)^2}}dx = \sin^{-1}\left(\frac{x-2}{2}\right) \qquad ...(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}\left(2\sqrt{4x-x^2}\right)+4\sin^{-1}\left(\frac{x-2}{2}\right)+C$$

$$= -\sqrt{4x-x^2}+4\sin^{-1}\left(\frac{x-2}{2}\right)+C$$

Question 21:

$$\frac{x+2}{\sqrt{x^2+2x+3}}$$

Answer

$$\int \frac{(x+2)}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\int \frac{2(x+2)}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \frac{1}{2}\int \frac{2}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

Let $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}dx$ and $I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$

$$\therefore \int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}I_1 + I_2 \quad ...(1)$$

Then, $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}dx$

Let $x^2 + 2x + 3 = t$

$\Rightarrow (2x + 2)\, dx = dt$

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2+2x+3} \qquad ...(2)$$

$$I_2 = \int \frac{1}{\sqrt{x^2+2x+3}} dx$$

$$\Rightarrow x^2+2x+3 = x^2+2x+1+2 = (x+1)^2+\left(\sqrt{2}\right)^2$$

$$\therefore I_2 = \int \frac{1}{\sqrt{(x+1)^2+\left(\sqrt{2}\right)^2}} dx = \log\left|(x+1)+\sqrt{x^2+2x+3}\right| \qquad ...(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x+2}{\sqrt{x^2+2x+3}} dx = \frac{1}{2}\left[2\sqrt{x^2+2x+3}\right] + \log\left|(x+1)+\sqrt{x^2+2x+3}\right| + C$$

$$= \sqrt{x^2+2x+3} + \log\left|(x+1)+\sqrt{x^2+2x+3}\right| + C$$

Question 22:

$$\frac{x+3}{x^2-2x-5}$$

Answer

$$\text{Let } (x+3) = A\frac{d}{dx}\left(x^2-2x-5\right)+B$$

$$(x+3) = A(2x-2)+B$$

Equating the coefficients of *x* and constant term on both sides, we obtain

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

$$-2A+B = 3 \Rightarrow B = 4$$

$$\therefore (x+3) = \frac{1}{2}(2x-2)+4$$

$$\Rightarrow \int \frac{x+3}{x^2-2x-5} dx = \int \frac{\frac{1}{2}(2x-2)+4}{x^2-2x-5} dx$$

$$= \frac{1}{2}\int \frac{2x-2}{x^2-2x-5} dx + 4\int \frac{1}{x^2-2x-5} dx$$

Let $I_1 = \int \frac{2x-2}{x^2-2x-5}dx$ and $I_2 = \int \frac{1}{x^2-2x-5}dx$

$\therefore \int \frac{x+3}{\left(x^2-2x-5\right)}dx = \frac{1}{2}I_1 + 4I_2$ ...(1)

Then, $I_1 = \int \frac{2x-2}{x^2-2x-5}dx$

Let $x^2-2x-5=t$

$\Rightarrow \left(2x-2\right)dx = dt$

$\Rightarrow I_1 = \int \frac{dt}{t} = \log|t| = \log\left|x^2-2x-5\right|$ ...(2)

$$I_2 = \int \frac{1}{x^2-2x-5}dx$$

$$= \int \frac{1}{\left(x^2-2x+1\right)-6}dx$$

$$= \int \frac{1}{(x-1)^2+\left(\sqrt{6}\right)^2}dx$$

$$= \frac{1}{2\sqrt{6}}\log\left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right) \quad ...(3)$$

Substituting (2) and (3) in (1), we obtain

$$\int \frac{x+3}{x^2-2x-5}dx = \frac{1}{2}\log\left|x^2-2x-5\right| + \frac{4}{2\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

$$= \frac{1}{2}\log\left|x^2-2x-5\right| + \frac{2}{\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

Question 23:

$$\frac{5x+3}{\sqrt{x^2+4x+10}}$$

Answer

Let $5x+3 = A\frac{d}{dx}\left(x^2+4x+10\right)+B$

$\Rightarrow 5x+3 = A(2x+4)+B$

Equating the coefficients of *x* and constant term, we obtain

$2A = 5 \Rightarrow A = \frac{5}{2}$

$4A+B = 3 \Rightarrow B = -7$

$\therefore 5x+3 = \frac{5}{2}(2x+4)-7$

$$\Rightarrow \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \int \frac{\frac{5}{2}(2x+4)-7}{\sqrt{x^2+4x+10}}dx$$

$$= \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}}dx - 7\int \frac{1}{\sqrt{x^2+4x+10}}dx$$

Let $I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}}dx$ and $I_2 = \int \frac{1}{\sqrt{x^2+4x+10}}dx$

$$\therefore \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}I_1 - 7I_2 \qquad ...(1)$$

Then, $I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}}dx$

Let $x^2+4x+10 = t$

$\therefore (2x+4)dx = dt$

$$\Rightarrow I_1 = \int \frac{dt}{t} = 2\sqrt{t} = 2\sqrt{x^2+4x+10} \qquad ...(2)$$

$$I_2 = \int \frac{1}{\sqrt{x^2+4x+10}}dx$$

$$= \int \frac{1}{\sqrt{\left(x^2+4x+4\right)+6}}dx$$

$$= \int \frac{1}{(x+2)^2+\left(\sqrt{6}\right)^2}dx$$

$$= \log\left|(x+2)\sqrt{x^2+4x+10}\right| \qquad ...(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}\left[2\sqrt{x^2+4x+10}\right] - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$
$$= 5\sqrt{x^2+4x+10} - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$

Question 24:

$\int \frac{dx}{x^2+2x+2}$ equals

A. $x \tan^{-1}(x + 1) + C$

B. $\tan^{-1}(x + 1) + C$

C. $(x + 1)\tan^{-1} x + C$

D. $\tan^{-1} x + C$

Answer

$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x^2+2x+1)+1}$$
$$= \int \frac{1}{(x+1)^2+(1)^2}dx$$
$$= \left[\tan^{-1}(x+1)\right] + C$$

Hence, the correct Answer is B.

Question 25:

$\int \frac{dx}{\sqrt{9x-4x^2}}$ equals

A. $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

B. $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$

C. $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

D. $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{9}\right)+C$

Answer

$$\int\frac{dx}{\sqrt{9x-4x^2}}$$

$$=\int\frac{1}{\sqrt{-4\left(x^2-\frac{9}{4}x\right)}}dx$$

$$=\int\frac{1}{-4\left(x^2-\frac{9}{4}x+\frac{81}{64}-\frac{81}{64}\right)}dx$$

$$=\int\frac{1}{\sqrt{-4\left[\left(x-\frac{9}{8}\right)^2-\left(\frac{9}{8}\right)^2\right]}}dx$$

$$=\frac{1}{2}\int\frac{1}{\sqrt{\left(\frac{9}{8}\right)^2-\left(x-\frac{9}{8}\right)^2}}dx$$

$$=\frac{1}{2}\left[\sin^{-1}\left(\frac{x-\frac{9}{8}}{\frac{9}{8}}\right)\right]+C \qquad \left(\int\frac{dy}{\sqrt{a^2-y^2}}=\sin^{-1}\frac{y}{a}+C\right)$$

$$=\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$$

Hence, the correct Answer is B.