Exercise 7.9

Question 1:

$$\int_{-1}^{1}(x+1)dx$$

Answer

$$\text{Let } I=\int_{-1}^{1}(x+1)dx$$

$$\int(x+1)\,dx=\frac{x^2}{2}+x=\text{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I=\text{F}(1)-\text{F}(-1)$$
$$=\left(\frac{1}{2}+1\right)-\left(\frac{1}{2}-1\right)$$
$$=\frac{1}{2}+1-\frac{1}{2}+1$$
$$=2$$

Question 2:

$$\int_{2}^{3}\frac{1}{x}dx$$

Answer

$$\text{Let } I=\int_{2}^{3}\frac{1}{x}\,dx$$

$$\int\frac{1}{x}\,dx=\log|x|=\text{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I=\text{F}(3)-\text{F}(2)$$
$$=\log|3|-\log|2|=\log\frac{3}{2}$$

Question 3:

$$\int_1^2 \left(4x^3 - 5x^2 + 6x + 9\right) dx$$

Answer

$$\text{Let } I = \int_1^2 \left(4x^3 - 5x^2 + 6x + 9\right) dx$$

$$\int \left(4x^3 - 5x^2 + 6x + 9\right) dx = 4\left(\frac{x^4}{4}\right) - 5\left(\frac{x^3}{3}\right) + 6\left(\frac{x^2}{2}\right) + 9(x)$$

$$= x^4 - \frac{5x^3}{3} + 3x^2 + 9x = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}(2) - \mathrm{F}(1)$$

$$I = \left\{2^4 - \frac{5\cdot(2)^3}{3} + 3(2)^2 + 9(2)\right\} - \left\{(1)^4 - \frac{5(1)^3}{3} + 3(1)^2 + 9(1)\right\}$$

$$= \left(16 - \frac{40}{3} + 12 + 18\right) - \left(1 - \frac{5}{3} + 3 + 9\right)$$

$$= 16 - \frac{40}{3} + 12 + 18 - 1 + \frac{5}{3} - 3 - 9$$

$$= 33 - \frac{35}{3}$$

$$= \frac{99 - 35}{3}$$

$$= \frac{64}{3}$$

Question 4:

$$\int_0^{\frac{\pi}{4}} \sin 2x dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \sin 2x \, dx$$

$$\int \sin 2x \, dx = \left(\frac{-\cos 2x}{2}\right) = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F\left(\frac{\pi}{4}\right) - F(0)$$

$$= -\frac{1}{2}\left[\cos 2\left(\frac{\pi}{4}\right) - \cos 0\right]$$

$$= -\frac{1}{2}\left[\cos\left(\frac{\pi}{2}\right) - \cos 0\right]$$

$$= -\frac{1}{2}[0-1]$$

$$= \frac{1}{2}$$

Question 5:

$$\int_0^{\frac{\pi}{2}} \cos 2x \, dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \cos 2x \, dx$$

$$\int \cos 2x \, dx = \left(\frac{\sin 2x}{2}\right) = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F\left(\frac{\pi}{2}\right) - F(0)$$

$$= \frac{1}{2}\left[\sin 2\left(\frac{\pi}{2}\right) - \sin 0\right]$$

$$= \frac{1}{2}[\sin\pi - \sin 0]$$

$$= \frac{1}{2}[0-0] = 0$$

Question 6:

$$\int_4^5 e^x \, dx$$

Answer

Let $I = \int_4^5 e^x dx$

$\int e^x dx = e^x = F(x)$

By second fundamental theorem of calculus, we obtain

$I = F(5) - F(4)$

$= e^5 - e^4$

$= e^4(e-1)$

Question 7:

$\int_0^{\frac{\pi}{4}} \tan x \, dx$

Answer

Let $I = \int_0^{\frac{\pi}{4}} \tan x \, dx$

$\int \tan x \, dx = -\log|\cos x| = F(x)$

By second fundamental theorem of calculus, we obtain

$I = F\left(\frac{\pi}{4}\right) - F(0)$

$= -\log\left|\cos\frac{\pi}{4}\right| + \log|\cos 0|$

$= -\log\left|\frac{1}{\sqrt{2}}\right| + \log|1|$

$= -\log(2)^{-\frac{1}{2}}$

$= \frac{1}{2}\log 2$

Question 8:

$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \; dx$

Answer

Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \, dx$

$\int \operatorname{cosec} x \; dx = \log\left|\operatorname{cosec} x - \cot x\right| = \mathrm{F}(x)$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}\left(\frac{\pi}{4}\right) - \mathrm{F}\left(\frac{\pi}{6}\right)$$

$$= \log\left|\operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4}\right| - \log\left|\operatorname{cosec}\frac{\pi}{6} - \cot\frac{\pi}{6}\right|$$

$$= \log\left|\sqrt{2} - 1\right| - \log\left|2 - \sqrt{3}\right|$$

$$= \log\left(\frac{\sqrt{2} - 1}{2 - \sqrt{3}}\right)$$

Question 9:

$$\int_0^1 \frac{dx}{\sqrt{1 - x^2}}$$

Answer

Let $I = \int_0^1 \frac{dx}{\sqrt{1 - x^2}}$

$\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x = \mathrm{F}(x)$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}(1) - \mathrm{F}(0)$$

$$= \sin^{-1}(1) - \sin^{-1}(0)$$

$$= \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2}$$

Question 10:

$$\int_0^1 \frac{dx}{1 + x^2}$$

Answer

$$\text{Let } I=\int_0^1 \frac{dx}{1+x^2}$$

$$\int \frac{dx}{1+x^2}=\tan^{-1} x=\mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned} I &= \mathrm{F}(1)-\mathrm{F}(0) \\ &= \tan^{-1}(1)-\tan^{-1}(0) \\ &= \frac{\pi}{4} \end{aligned}$$

Question 11:

$$\int_2^3 \frac{dx}{x^2-1}$$

Answer

$$\text{Let } I=\int_2^3 \frac{dx}{x^2-1}$$

$$\int \frac{dx}{x^2-1}=\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|=\mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned} I &= \mathrm{F}(3)-\mathrm{F}(2) \\ &= \frac{1}{2}\left[\log\left|\frac{3-1}{3+1}\right|-\log\left|\frac{2-1}{2+1}\right|\right] \\ &= \frac{1}{2}\left[\log\left|\frac{2}{4}\right|-\log\left|\frac{1}{3}\right|\right] \\ &= \frac{1}{2}\left[\log\frac{1}{2}-\log\frac{1}{3}\right] \\ &= \frac{1}{2}\left[\log\frac{3}{2}\right] \end{aligned}$$

Question 12:

$$\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \cos^2 x\, dx$$

$$\int \cos^2 x\, dx = \int \left(\frac{1+\cos 2x}{2}\right) dx = \frac{x}{2} + \frac{\sin 2x}{4} = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = \left[\mathrm{F}\left(\frac{\pi}{2}\right) - \mathrm{F}(0)\right]$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - \frac{\sin \pi}{2}\right) - \left(0 + \frac{\sin 0}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + 0 - 0 - 0\right]$$

$$= \frac{\pi}{4}$$

Question 13:

$$\int_2^3 \frac{x\,dx}{x^2+1}$$

Answer

$$\text{Let } I = \int_2^3 \frac{x}{x^2+1} dx$$

$$\int \frac{x}{x^2+1} dx = \frac{1}{2}\int \frac{2x}{x^2+1} dx = \frac{1}{2}\log\left(1+x^2\right) = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}(3) - \mathrm{F}(2)$$

$$= \frac{1}{2}\left[\log\left(1+(3)^2\right) - \log\left(1+(2)^2\right)\right]$$

$$= \frac{1}{2}\left[\log(10) - \log(5)\right]$$

$$= \frac{1}{2}\log\left(\frac{10}{5}\right) = \frac{1}{2}\log 2$$

Question 14:

$$\int_0^1 \frac{2x+3}{5x^2+1}dx$$

Answer

$$\text{Let } I = \int_0^1 \frac{2x+3}{5x^2+1}dx$$

$$\int \frac{2x+3}{5x^2+1}dx = \frac{1}{5}\int \frac{5(2x+3)}{5x^2+1}dx$$

$$= \frac{1}{5}\int \frac{10x+15}{5x^2+1}dx$$

$$= \frac{1}{5}\int \frac{10x}{5x^2+1}dx + 3\int \frac{1}{5x^2+1}dx$$

$$= \frac{1}{5}\int \frac{10x}{5x^2+1}dx + 3\int \frac{1}{5\left(x^2+\frac{1}{5}\right)}dx$$

$$= \frac{1}{5}\log\left(5x^2+1\right) + \frac{3}{5}\cdot\frac{1}{\frac{1}{\sqrt{5}}}\tan^{-1}\frac{x}{\frac{1}{\sqrt{5}}}$$

$$= \frac{1}{5}\log\left(5x^2+1\right) + \frac{3}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}x\right)$$

$$= \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}(1) - \mathrm{F}(0)$$

$$= \left\{\frac{1}{5}\log(5+1) + \frac{3}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}\right)\right\} - \left\{\frac{1}{5}\log(1) + \frac{3}{\sqrt{5}}\tan^{-1}(0)\right\}$$

$$= \frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$$

Question 15:

$$\int_0^1 xe^{x^2}dx$$

Answer

Let $I=\int_0^1 xe^{x^2}dx$

Put $x^2=t \Rightarrow\ 2x\ dx=dt$

As $x\to 0, t\to 0$ and as $x\to 1, t\to 1$,

$\therefore I=\frac{1}{2}\int_0^1 e^t dt$

$\frac{1}{2}\int e^t dt=\frac{1}{2}e^t=\mathrm{F}(t)$

By second fundamental theorem of calculus, we obtain

$I=\mathrm{F}(1)-\mathrm{F}(0)$

$=\frac{1}{2}e-\frac{1}{2}e^0$

$=\frac{1}{2}(e-1)$

Question 16:

$$\int_0^1 \frac{5x^2}{x^2+4x+3}$$

Answer

Let $I=\int_1^2 \frac{5x^2}{x^2+4x+3}dx$

Dividing $5x^2$ by x^2+4x+3, we obtain

$I=\int_1^2\left\{5-\frac{20x+15}{x^2+4x+3}\right\}dx$

$=\int_1^2 5dx-\int_1^2\frac{20x+15}{x^2+4x+3}dx$

$=[5x]_1^2-\int_1^2\frac{20x+15}{x^2+4x+3}dx$

$I=5-I_1$, where $I=\int_1^2\frac{20x+15}{x^2+4x+3}dx$ ...(1)

Consider $I_1 = \int_1^2 \frac{20x+15}{x^2+4x+8}dx$

Let $20x+15 = A\frac{d}{dx}\left(x^2+4x+3\right)+B$

$= 2Ax + \left(4A+B\right)$

Equating the coefficients of x and constant term, we obtain

A = 10 and B = −25

$\Rightarrow I_1 = 10\int_1^2 \frac{2x+4}{x^2+4x+3}dx - 25\int_1^2 \frac{dx}{x^2+4x+3}$

Let $x^2+4x+3 = t$

$\Rightarrow \left(2x+4\right)dx = dt$

$$\Rightarrow I_1 = 10\int \frac{dt}{t} - 25\int \frac{dx}{\left(x+2\right)^2 - 1^2}$$

$$= 10\log t - 25\left[\frac{1}{2}\log\left(\frac{x+2-1}{x+2+1}\right)\right]$$

$$= \left[10\log\left(x^2+4x+3\right)\right]_1^2 - 25\left[\frac{1}{2}\log\left(\frac{x+1}{x+3}\right)\right]_1^2$$

$$= \left[10\log 15 - 10\log 8\right] - 25\left[\frac{1}{2}\log\frac{3}{5} - \frac{1}{2}\log\frac{2}{4}\right]$$

$$= \left[10\log\left(5\times 3\right) - 10\log\left(4\times 2\right)\right] - \frac{25}{2}\left[\log 3 - \log 5 - \log 2 + \log 4\right]$$

$$= \left[10\log 5 + 10\log 3 - 10\log 4 - 10\log 2\right] - \frac{25}{2}\left[\log 3 - \log 5 - \log 2 + \log 4\right]$$

$$= \left[10 + \frac{25}{2}\right]\log 5 + \left[-10 - \frac{25}{2}\right]\log 4 + \left[10 - \frac{25}{2}\right]\log 3 + \left[-10 + \frac{25}{2}\right]\log 2$$

$$= \frac{45}{2}\log 5 - \frac{45}{2}\log 4 - \frac{5}{2}\log 3 + \frac{5}{2}\log 2$$

$$= \frac{45}{2}\log\frac{5}{4} - \frac{5}{2}\log\frac{3}{2}$$

Substituting the value of I_1 in (1), we obtain

$$I = 5 - \left[\frac{45}{2}\log\frac{5}{4} - \frac{5}{2}\log\frac{3}{2}\right]$$

$$= 5 - \frac{5}{2}\left[9\log\frac{5}{4} - \log\frac{3}{2}\right]$$

Question 17:

$$\int_0^{\frac{\pi}{4}}\left(2\sec^2 x + x^3 + 2\right)dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{4}}\left(2\sec^2 x + x^3 + 2\right)dx$$

$$\int\left(2\sec^2 x + x^3 + 2\right)dx = 2\tan x + \frac{x^4}{4} + 2x = \text{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = \text{F}\left(\frac{\pi}{4}\right) - \text{F}(0)$$

$$= \left\{\left(2\tan\frac{\pi}{4} + \frac{1}{4}\left(\frac{\pi}{4}\right)^4 + 2\left(\frac{\pi}{4}\right)\right) - \left(2\tan 0 + 0 + 0\right)\right\}$$

$$= 2\tan\frac{\pi}{4} + \frac{\pi^4}{4^5} + \frac{\pi}{2}$$

$$= 2 + \frac{\pi}{2} + \frac{\pi^4}{1024}$$

Question 18:

$$\int_0^{\pi}\left(\sin^2\frac{x}{2} - \cos^2\frac{x}{2}\right)dx$$

Answer

$$\text{Let } I = \int_0^{\pi} \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) dx$$

$$= -\int_0^{\pi} \left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \right) dx$$

$$= -\int_0^{\pi} \cos x \; dx$$

$$\int \cos x \; dx = \sin x = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}(\pi) - \mathrm{F}(0)$$

$$= \sin\pi - \sin 0$$

$$= 0$$

Question 19:

$$\int_0^2 \frac{6x+3}{x^2+4} dx$$

Answer

$$\text{Let } I = \int_0^2 \frac{6x+3}{x^2+4} dx$$

$$\int \frac{6x+3}{x^2+4} dx = 3\int \frac{2x+1}{x^2+4} dx$$

$$= 3\int \frac{2x}{x^2+4} dx + 3\int \frac{1}{x^2+4} dx$$

$$= 3\log\left(x^2+4\right) + \frac{3}{2}\tan^{-1}\frac{x}{2} = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(2) - F(0)$$

$$= \left\{3\log\left(2^2+4\right)+\frac{3}{2}\tan^{-1}\left(\frac{2}{2}\right)\right\} - \left\{3\log\left(0+4\right)+\frac{3}{2}\tan^{-1}\left(\frac{0}{2}\right)\right\}$$

$$= 3\log 8 + \frac{3}{2}\tan^{-1}1 - 3\log 4 - \frac{3}{2}\tan^{-1}0$$

$$= 3\log 8 + \frac{3}{2}\left(\frac{\pi}{4}\right) - 3\log 4 - 0$$

$$= 3\log\left(\frac{8}{4}\right) + \frac{3\pi}{8}$$

$$= 3\log 2 + \frac{3\pi}{8}$$

Question 20:

$$\int_0^1\left(xe^x + \sin\frac{\pi x}{4}\right)dx$$

Answer

$$\text{Let } I = \int_0^1\left(xe^x + \sin\frac{\pi x}{4}\right)dx$$

$$\int\left(xe^x + \sin\frac{\pi x}{4}\right)dx = x\int e^x dx - \int\left\{\left(\frac{d}{dx}x\right)\int e^x dx\right\}dx + \left\{\frac{-\cos\frac{\pi x}{4}}{\frac{\pi}{4}}\right\}$$

$$= xe^x - \int e^x dx - \frac{4\pi}{\pi}\cos\frac{x}{4}$$

$$= xe^x - e^x - \frac{4\pi}{\pi}\cos\frac{x}{4}$$

$$= F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(1) - F(0)$$

$$= \left(1.e^1 - e^1 - \frac{4}{\pi}\cos\frac{\pi}{4}\right) - \left(0.e^0 - e^0 - \frac{4}{\pi}\cos 0\right)$$

$$= e - e - \frac{4}{\pi}\left(\frac{1}{\sqrt{2}}\right) + 1 + \frac{4}{\pi}$$

$$= 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}$$

Question 21:

$\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$ equals

A. $\frac{\pi}{3}$

B. $\frac{2\pi}{3}$

C. $\frac{\pi}{6}$

D. $\frac{\pi}{12}$

Answer

$$\int \frac{dx}{1+x^2} = \tan^{-1} x = F(x)$$

By second fundamental theorem of calculus, we obtain

$$\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = F(\sqrt{3}) - F(1)$$

$$= \tan^{-1}\sqrt{3} - \tan^{-1} 1$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

Hence, the correct Answer is D.

Question 22:

$\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2}$ equals

A. $\frac{\pi}{6}$

B. $\frac{\pi}{12}$

C. $\frac{\pi}{24}$

D. $\frac{\pi}{4}$

Answer

$$\int \frac{dx}{4+9x^2} = \int \frac{dx}{(2)^2+(3x)^2}$$

Put $3x = t \Rightarrow 3dx = dt$

$$\therefore \int \frac{dx}{(2)^2+(3x)^2} = \frac{1}{3}\int \frac{dt}{(2)^2+t^2}$$

$$= \frac{1}{3}\left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right)$$

$$= \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2} = F\left(\frac{2}{3}\right) - F(0)$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3}{2}\cdot\frac{2}{3}\right) - \frac{1}{6}\tan^{-1} 0$$

$$= \frac{1}{6}\tan^{-1} 1 - 0$$

$$= \frac{1}{6}\times\frac{\pi}{4}$$

$$= \frac{\pi}{24}$$

Hence, the correct Answer is C.