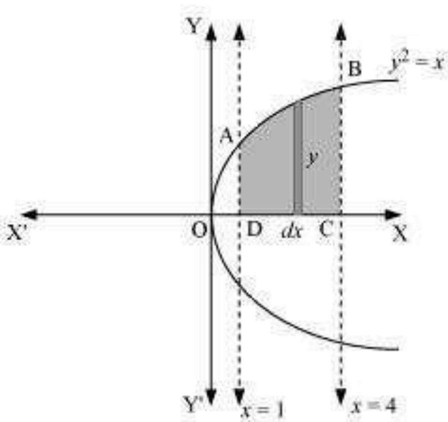
Exercise 8.1

Question 1:

Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1$, $x = 4$ and the x-axis.

Answer



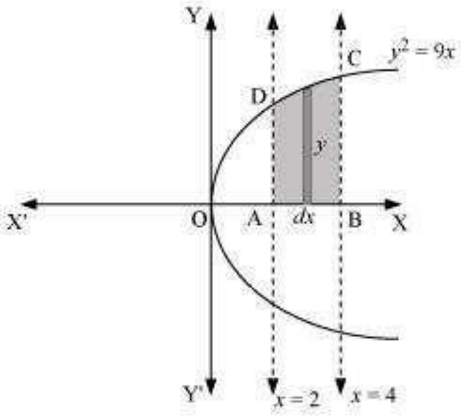
The area of the region bounded by the curve, $y^2 = x$, the lines, $x = 1$ and $x = 4$, and the x-axis is the area ABCD.

$$\text{Area of ABCD} = \int_1^4 y\,dx$$
$$= \int_1^4 \sqrt{x}\,dx$$
$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$$
$$= \frac{2}{3}\left[(4)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right]$$
$$= \frac{2}{3}[8-1]$$
$$= \frac{14}{3} \text{ units}$$

Question 2:

Find the area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$ and the x-axis in the first quadrant.

Answer



The area of the region bounded by the curve, $y^2 = 9x$, $x = 2$, and $x = 4$, and the x-axis is the area ABCD.

$$\text{Area of ABCD} = \int_2^4 y\,dx$$

$$= \int_2^4 3\sqrt{x}\,dx$$

$$= 3\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^4$$

$$= 2\left[x^{\frac{3}{2}}\right]_2^4$$

$$= 2\left[(4)^{\frac{3}{2}} - (2)^{\frac{3}{2}}\right]$$

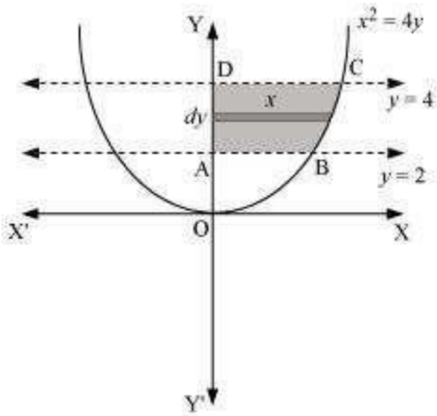
$$= 2\left[8 - 2\sqrt{2}\right]$$

$$= \left(16 - 4\sqrt{2}\right) \text{ units}$$

Question 3:

Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the y-axis in the first quadrant.

Answer



The area of the region bounded by the curve, $x^2 = 4y$, $y = 2$, and $y = 4$, and the y-axis is the area ABCD.

$$\text{Area of ABCD} = \int_2^4 x\,dy$$

$$= \int_2^4 2\sqrt{y}\,dy$$

$$= 2\int_2^4 \sqrt{y}\,dy$$

$$= 2\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^4$$

$$= \frac{4}{3}\left[(4)^{\frac{3}{2}} - (2)^{\frac{3}{2}}\right]$$

$$= \frac{4}{3}\left[8 - 2\sqrt{2}\right]$$

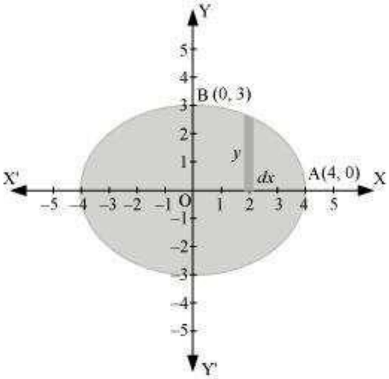
$$= \left(\frac{32 - 8\sqrt{2}}{3}\right) \text{ units}$$

Question 4:

Find the area of the region bounded by the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$

Answer

The given equation of the ellipse, $\frac{x^2}{16}+\frac{y^2}{9}=1$, can be represented as



It can be observed that the ellipse is symmetrical about *x*-axis and *y*-axis.

∴ Area bounded by ellipse = 4 × Area of OAB

$$\text{Area of OAB} = \int_0^4 y\,dx$$
$$= \int_0^4 3\sqrt{1-\frac{x^2}{16}}dx$$
$$= \frac{3}{4}\int_0^4 \sqrt{16-x^2}dx$$
$$= \frac{3}{4}\left[\frac{x}{2}\sqrt{16-x^2}+\frac{16}{2}\sin^{-1}\frac{x}{4}\right]_0^4$$
$$= \frac{3}{4}\left[2\sqrt{16-16}+8\sin^{-1}(1)-0-8\sin^{-1}(0)\right]$$
$$= \frac{3}{4}\left[\frac{8\pi}{2}\right]$$
$$= \frac{3}{4}[4\pi]$$
$$= 3\pi$$

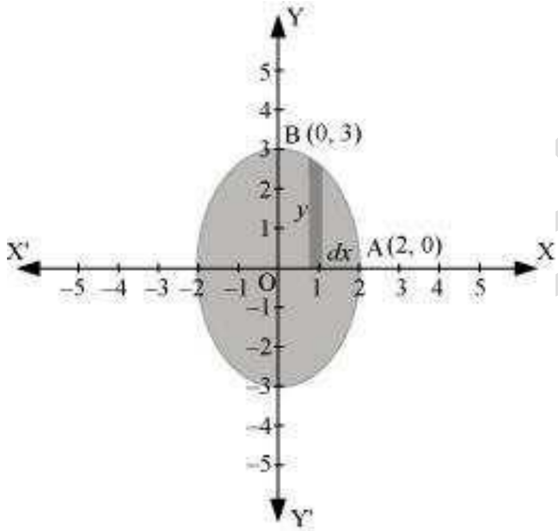
Therefore, area bounded by the ellipse = 4 × 3π = 12π units

Question 5:

Find the area of the region bounded by the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$

Answer

The given equation of the ellipse can be represented as



$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

$$\Rightarrow y=3\sqrt{1-\frac{x^2}{4}} \qquad ...(1)$$

It can be observed that the ellipse is symmetrical about *x*-axis and *y*-axis.

∴ Area bounded by ellipse = 4 × Area OAB

$\therefore$ Area of OAB $= \int_0^2 y\,dx$

$$= \int_0^2 3\sqrt{1-\frac{x^2}{4}}\,dx \qquad \text{[Using (1)]}$$

$$= \frac{3}{2}\int_0^2 \sqrt{4-x^2}\,dx$$

$$= \frac{3}{2}\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= \frac{3}{2}\left[\frac{2\pi}{2}\right]$$

$$= \frac{3\pi}{2}$$

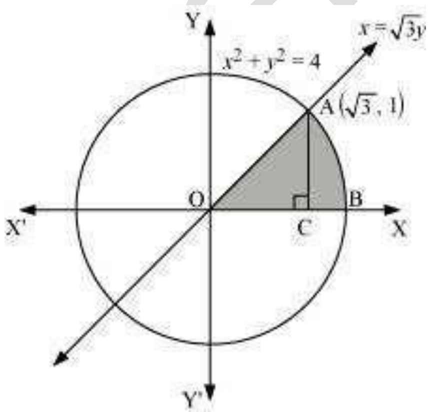
Therefore, area bounded by the ellipse = $4\times\frac{3\pi}{2} = 6\pi$ units

Question 6:

Find the area of the region in the first quadrant enclosed by *x*-axis, line $x=\sqrt{3}y$ and the circle $x^2+y^2=4$

Answer

The area of the region bounded by the circle, $x^2+y^2=4, x=\sqrt{3}y$, and the *x*-axis is the area OAB.



The point of intersection of the line and the circle in the first quadrant is $\left(\sqrt{3},1\right)$.

Area OAB = Area ΔOCA + Area ACB

Area of OAC $=\frac{1}{2}\times \text{OC}\times \text{AC}=\frac{1}{2}\times\sqrt{3}\times 1=\frac{\sqrt{3}}{2}$ ...(1)

Area of ABC $=\int_{\sqrt{3}}^{2} y\,dx$

$$=\int_{\sqrt{3}}^{2}\sqrt{4-x^2}\,dx$$

$$=\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{\sqrt{3}}^{2}$$

$$=\left[2\times\frac{\pi}{2}-\frac{\sqrt{3}}{2}\sqrt{4-3}-2\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$=\left[\pi-\frac{\sqrt{3}\pi}{2}-2\left(\frac{\quad}{3}\right)\right]$$

$$=\left[\pi-\frac{\sqrt{3}}{2}-\frac{2\pi}{3}\right]$$

$$=\left[\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right] \qquad ...(2)$$

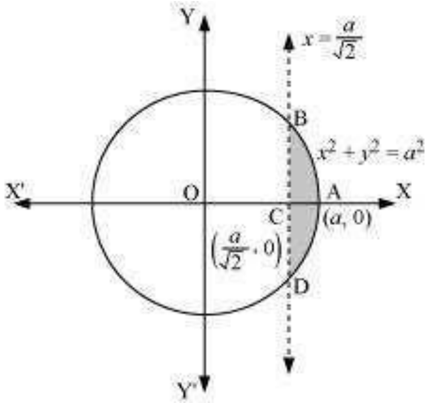
Therefore, area enclosed by x-axis, the line $x=\sqrt{3}y$, and the circle $x^2+y^2=4$ in the first quadrant = $\frac{\sqrt{3}\pi}{2}+\frac{\quad}{3}-\frac{3\sqrt{\quad}\pi}{2}=\frac{\quad}{3}$ units

Question 7:

Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x=\frac{a}{\sqrt{2}}$

Answer

The area of the smaller part of the circle, $x^2 + y^2 = a^2$, cut off by the line, $x = \frac{a}{\sqrt{2}}$, is the area ABCDA.



It can be observed that the area ABCD is symmetrical about x-axis.

∴ Area ABCD = 2 × Area ABC

$$Area\ of\ ABC = \int_{\frac{a}{\sqrt{2}}}^{a} y\,dx$$

$$= \int_{\frac{a}{\sqrt{2}}}^{a} \sqrt{a^2 - x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{\frac{a}{\sqrt{2}}}^{a}$$

$$= \left[\frac{a^2}{2}\left(\frac{\pi}{2}\right) - \frac{a}{2\sqrt{2}}\sqrt{a^2 - \frac{a^2}{2}} - \frac{a^2}{2}\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right]$$

$$= \frac{a^2\pi}{4} - \frac{a}{2\sqrt{2}}\cdot\frac{a}{\sqrt{2}} - \frac{a^2}{2}\left(\frac{\pi}{4}\right)$$

$$= \frac{a^2\pi}{4} - \frac{a^2}{4} - \frac{a^2\pi}{8}$$

$$= \frac{a^2}{4}\left[\pi - 1 - \frac{\pi}{2}\right]$$

$$= \frac{a^2}{4}\left[\frac{\pi}{2} - 1\right]$$

$$\Rightarrow Area\ ABCD = 2\left[\frac{a^2}{4}\left(\frac{\pi}{2} - 1\right)\right] = \frac{a^2}{2}\left(\frac{\pi}{2} - 1\right)$$

Therefore, the area of smaller part of the circle, $x^2 + y^2 = a^2$, cut off by the line, $x = \frac{a}{\sqrt{2}}$, is $\frac{a^2}{2}\left(\frac{\pi}{2} - 1\right)$ units.

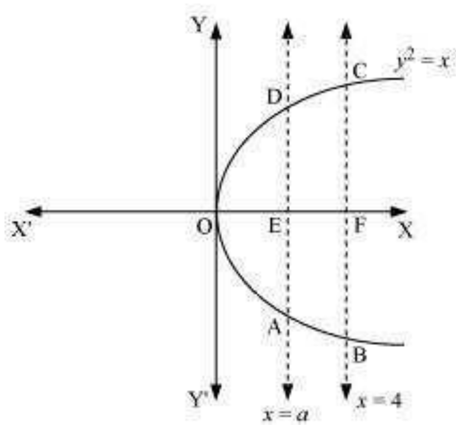
Question 8:

The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.

Answer

The line, $x = a$, divides the area bounded by the parabola and $x = 4$ into two equal parts.

∴ Area OAD = Area ABCD



It can be observed that the given area is symmetrical about *x*-axis.

⇒ Area OED = Area EFCD

$$Area\ OED = \int_0^a y\,dx$$
$$= \int_0^a \sqrt{x}\,dx$$
$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^a$$
$$= \frac{2}{3}(a)^{\frac{3}{2}} \qquad ...(1)$$

$$Area\ of\ EFCD = \int_0^4 \sqrt{x}dx$$
$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$
$$= \frac{2}{3}\left[8 - a^{\frac{3}{2}}\right] \qquad ...(2)$$

From (1) and (2), we obtain

$$\frac{2}{3}(a)^{\frac{3}{2}} = \frac{2}{3}\left[8 - (a)^{\frac{3}{2}}\right]$$
$$\Rightarrow 2\cdot(a)^{\frac{3}{2}} = 8$$
$$\Rightarrow (a)^{\frac{3}{2}} = 4$$
$$\Rightarrow a = (4)^{\frac{2}{3}}$$

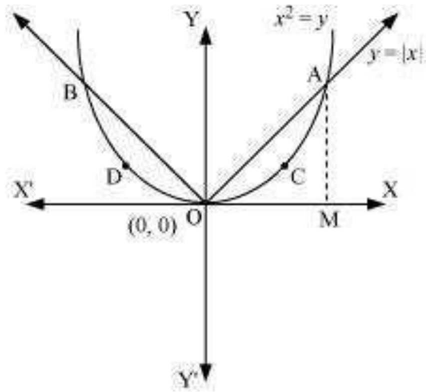
Therefore, the value of *a* is $(4)^{\frac{2}{3}}$.

Question 9:

Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$

Answer

The area bounded by the parabola, $x^2 = y$,and the line, $y = |x|$, can be represented as



The given area is symmetrical about *y*-axis.

∴ Area OACO = Area ODBO

The point of intersection of parabola, $x^2 = y$, and line, $y = x$, is A (1, 1).

Area of OACO = Area ΔOAB – Area OBACO

$$\therefore \text{Area of } \Delta OAB = \frac{1}{2} \times OB \times AB = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$$

$$\text{Area of OBACO} = \int_0^1 y\,dx = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

⇒ Area of OACO = Area of ΔOAB – Area of OBACO

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6}$$

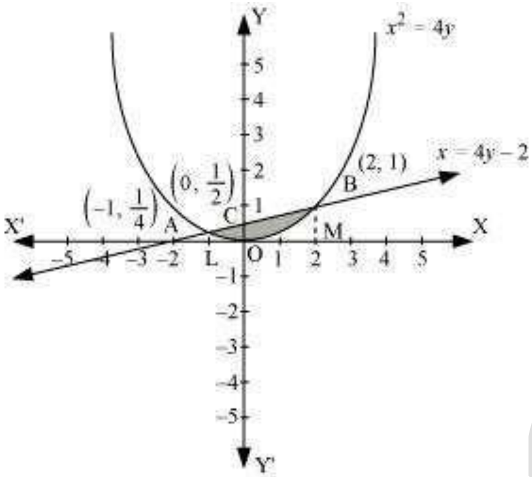
Therefore, required area = $2\left[\frac{1}{6}\right] = \frac{1}{3}$ units

Question 10:

Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$

Answer

The area bounded by the curve, $x^2 = 4y$, and line, $x = 4y - 2$, is represented by the shaded area OBAO.



Let A and B be the points of intersection of the line and parabola.

Coordinates of point A are $\left(-1, \frac{1}{4}\right)$.

Coordinates of point B are (2, 1).

We draw AL and BM perpendicular to x-axis.

It can be observed that,

Area OBAO = Area OBCO + Area OACO … (1)

Then, Area OBCO = Area OMBC – Area OMBO

$$= \int_0^2 \frac{x+2}{4}dx - \int_0^2 \frac{x^2}{4}dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2}+2x\right]_0^2 - \frac{1}{4}\left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{4}[2+4] - \frac{1}{4}\left[\frac{8}{3}\right]$$

$$= \frac{3}{2} - \frac{2}{3}$$

$$= \frac{5}{6}$$

Similarly, Area OACO = Area OLAC – Area OLAO

$$= \int_{-1}^0 \frac{x+2}{4}dx - \int_{-1}^0 \frac{x^2}{4}dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2}+2x\right]_{-1}^0 - \frac{1}{4}\left[\frac{x^3}{3}\right]_{-1}^0$$

$$= -\frac{1}{4}\left[\frac{(-1)^2}{2}+2(-1)\right] - \left[-\frac{1}{4}\left(\frac{(-1)^3}{3}\right)\right]$$

$$= -\frac{1}{4}\left[\frac{1}{2}-2\right] - \frac{1}{12}$$

$$= \frac{1}{2} - \frac{1}{8} - \frac{1}{12}$$

$$= \frac{7}{24}$$

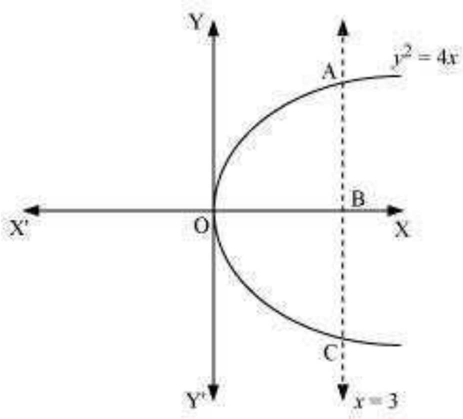
Therefore, required area = $\left(\frac{5}{6}+\frac{7}{24}\right) = \frac{9}{8}$ units

Question 11:

Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$

Answer

The region bounded by the parabola, $y^2 = 4x$, and the line, $x = 3$, is the area OACO.



The area OACO is symmetrical about x-axis.

∴ Area of OACO = 2 (Area of OAB)

$$\text{Area OACO} = 2\left[\int_0^3 y\,dx\right]$$
$$= 2\int_0^3 2\sqrt{x}\,dx$$
$$= 4\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3$$
$$= \frac{8}{3}\left[(3)^{\frac{3}{2}}\right]$$
$$= 8\sqrt{3}$$

Therefore, the required area is $8\sqrt{3}$ units.

Question 12:

Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is

A. π

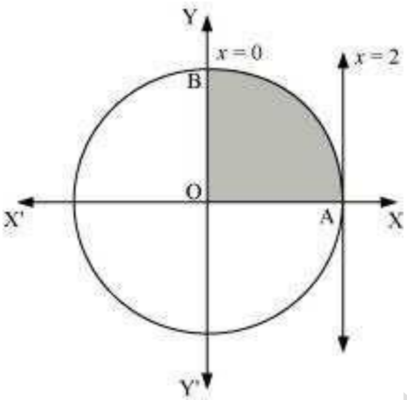
B. $\frac{\pi}{2}$

C. $\frac{\pi}{3}$

D. $\frac{\pi}{4}$

Answer

The area bounded by the circle and the lines, $x = 0$ and $x = 2$, in the first quadrant is represented as



$$\therefore \text{Area OAB} = \int_0^2 y\,dx$$

$$= \int_0^2 \sqrt{4-x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 2\left(\frac{\pi}{2}\right)$$

$$= \pi \text{ units}$$

Thus, the correct answer is A.

Question 13:

Area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$ is

A. 2

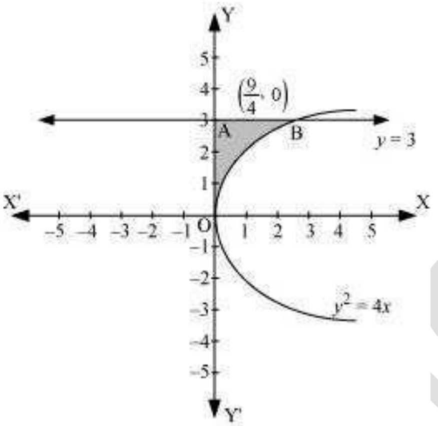
B. $\frac{9}{4}$

C. $\frac{9}{3}$

D. $\frac{9}{2}$

Answer

The area bounded by the curve, $y^2 = 4x$, y-axis, and $y = 3$ is represented as



$$\therefore \text{Area OAB} = \int_0^3 x\,dy$$

$$= \int_0^3 \frac{y^2}{4}dy$$

$$= \frac{1}{4}\left[\frac{y^3}{3}\right]_0^3$$

$$= \frac{1}{12}(27)$$

$$= \frac{9}{4} \text{ units}$$

Thus, the correct answer is B.