

Exercise 9.5

Question 1:

$$(x^2 + xy)dy = (x^2 + y^2)dx$$

Answer

The given differential equation i.e., $(x^2 + xy) dy = (x^2 + y^2) dx$ can be written as:

$$\frac{dy}{dx} = \frac{x^2 + y^2}{x^2 + xy} \quad \dots(1)$$

$$\text{Let } F(x, y) = \frac{x^2 + y^2}{x^2 + xy}.$$

$$\text{Now, } F(\lambda x, \lambda y) = \frac{(\lambda x)^2 + (\lambda y)^2}{(\lambda x)^2 + (\lambda x)(\lambda y)} = \frac{x^2 + y^2}{x^2 + xy} = \lambda^0 \cdot F(x, y)$$

This shows that equation (1) is a homogeneous equation.

To solve it, we make the substitution as:

$$y = vx$$

Differentiating both sides with respect to x , we get:

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

Substituting the values of v and $\frac{dy}{dx}$ in equation (1), we get:

$$\begin{aligned}
v+x \frac{dv}{dx} &= \frac{x^2 + (vx)^2}{x^2 + x(vx)} \\
\Rightarrow v+x \frac{dv}{dx} &= \frac{1+v^2}{1+v} \\
\Rightarrow x \frac{dv}{dx} &= \frac{1+v^2}{1+v} - v = \frac{(1+v^2) - v(1+v)}{1+v} \\
\Rightarrow x \frac{dv}{dx} &= \frac{1-v}{1+v} \\
\Rightarrow \left(\frac{1+v}{1-v} \right) &= dv = \frac{dx}{x} \\
\Rightarrow \left(\frac{2-1+v}{1-v} \right) dv &= \frac{dx}{x} \\
\Rightarrow \left(\frac{2}{1-v} - 1 \right) dv &= \frac{dx}{x}
\end{aligned}$$

Integrating both sides, we get:

$$\begin{aligned}
-2 \log(1-v) - v &= \log x - \log k \\
\Rightarrow v &= -2 \log(1-v) - \log x + \log k \\
\Rightarrow v &= \log \left[\frac{k}{x(1-v)^2} \right] \\
\Rightarrow \frac{y}{x} &= \log \left[\frac{k}{x \left(1 - \frac{y}{x} \right)^2} \right] \\
\Rightarrow \frac{y}{x} &= \log \left[\frac{kx}{(x-y)^2} \right] \\
\Rightarrow \frac{kx}{(x-y)^2} &= e^{\frac{y}{x}} \\
\Rightarrow (x-y)^2 &= kx e^{-\frac{y}{x}}
\end{aligned}$$

This is the required solution of the given differential equation.

Question 2:

$$y' = \frac{x+y}{x}$$

Answer

The given differential equation is:

$$y' = \frac{x+y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x} \quad \dots(1)$$

$$\text{Let } F(x, y) = \frac{x+y}{x}.$$

$$\text{Now, } F(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\lambda x} = \frac{x+y}{x} = \lambda^0 F(x, y)$$

Thus, the given equation is a homogeneous equation.

To solve it, we make the substitution as:

$$y = vx$$

Differentiating both sides with respect to x , we get:

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x \frac{dv}{dx} = \frac{x+vx}{x}$$

$$\Rightarrow v + x \frac{dv}{dx} = 1 + v$$

$$x \frac{dv}{dx} = 1$$

$$\Rightarrow dv = \frac{dx}{x}$$

Integrating both sides, we get:

$$v = \log x + C$$

$$\Rightarrow \frac{y}{x} = \log x + C$$

$$\Rightarrow y = x \log x + Cx$$

This is the required solution of the given differential equation.

Question 3:

$$(x - y)dy - (x + y)dx = 0$$

Answer

The given differential equation is:

$$(x - y)dy - (x + y)dx = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x + y}{x - y} \quad \dots(1)$$

$$\text{Let } F(x, y) = \frac{x + y}{x - y}.$$

$$\therefore F(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\lambda x - \lambda y} = \frac{x + y}{x - y} = \lambda^0 \cdot F(x, y)$$

Thus, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$$y = vx$$

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x \frac{dv}{dx} = \frac{x + vx}{x - vx} = \frac{1 + v}{1 - v}$$

$$x \frac{dv}{dx} = \frac{1 + v}{1 - v} - v = \frac{1 + v - v(1 - v)}{1 - v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{1 + v^2}{1 - v}$$

$$\Rightarrow \frac{1 - v}{(1 + v^2)} dv = \frac{dx}{x}$$

$$\Rightarrow \left(\frac{1}{1 + v^2} - \frac{v}{1 - v^2} \right) dv = \frac{dx}{x}$$

Integrating both sides, we get:

$$\tan^{-1} v - \frac{1}{2} \log(1 + v^2) = \log x + C$$

$$\Rightarrow \tan^{-1} \left(\frac{y}{x} \right) - \frac{1}{2} \log \left[1 + \left(\frac{y}{x} \right)^2 \right] = \log x + C$$

$$\Rightarrow \tan^{-1} \left(\frac{y}{x} \right) - \frac{1}{2} \log \left(\frac{x^2 + y^2}{x^2} \right) = \log x + C$$

$$\Rightarrow \tan^{-1} \left(\frac{y}{x} \right) - \frac{1}{2} [\log(x^2 + y^2) - \log x^2] = \log x + C$$

$$\Rightarrow \tan^{-1} \left(\frac{y}{x} \right) = \frac{1}{2} \log(x^2 + y^2) + C$$

This is the required solution of the given differential equation.

Question 4:

$$(x^2 - y^2) dx + 2xy dy = 0$$

Answer

The given differential equation is:

$$(x^2 - y^2) dx + 2xy dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(x^2 - y^2)}{2xy} \quad \dots(1)$$

$$\text{Let } F(x, y) = \frac{-(x^2 - y^2)}{2xy}.$$

$$\therefore F(\lambda x, \lambda y) = \left[\frac{(\lambda x)^2 - (\lambda y)^2}{2(\lambda x)(\lambda y)} \right] = \frac{-(x^2 - y^2)}{2xy} = \lambda^0 \cdot F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$$y = vx$$

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x \frac{dv}{dx} = - \left[\frac{x^2 - (vx)^2}{2x \cdot (vx)} \right]$$

$$v + x \frac{dv}{dx} = \frac{v^2 - 1}{2v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v^2 - 1}{2v} - v = \frac{v^2 - 1 - 2v^2}{2v}$$

$$\Rightarrow x \frac{dv}{dx} = - \frac{(1 + v^2)}{2v}$$

$$\Rightarrow \frac{2v}{1 + v^2} dv = - \frac{dx}{x}$$

Integrating both sides, we get:

$$\log(1 + v^2) = -\log x + \log C = \log \frac{C}{x}$$

$$\Rightarrow 1 + v^2 = \frac{C}{x}$$

$$\Rightarrow \left[1 + \frac{y^2}{x^2} \right] = \frac{C}{x}$$

$$\Rightarrow x^2 + y^2 = Cx$$

This is the required solution of the given differential equation.

Question 5:

$$x^2 \frac{dy}{dx} - x^2 - 2y^2 + xy$$

Answer

The given differential equation is:

$$x^2 \frac{dy}{dx} = x^2 - 2y^2 + xy$$

$$\frac{dy}{dx} = \frac{x^2 - 2y^2 + xy}{x^2} \quad \dots(1)$$

$$\text{Let } F(x, y) = \frac{x^2 - 2y^2 + xy}{x^2}.$$

$$\therefore F(\lambda x, \lambda y) = \frac{(\lambda x)^2 - 2(\lambda y)^2 + (\lambda x)(\lambda y)}{(\lambda x)^2} = \frac{x^2 - 2y^2 + xy}{x^2} = \lambda^0 \cdot F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$$y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x \frac{dv}{dx} = \frac{x^2 - 2(vx)^2 + x \cdot (vx)}{x^2}$$

$$\Rightarrow v + x \frac{dv}{dx} = 1 - 2v^2 + v$$

$$\Rightarrow x \frac{dv}{dx} = 1 - 2v^2$$

$$\Rightarrow \frac{dv}{1 - 2v^2} = \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2} \cdot \frac{dv}{\frac{1}{2} - v^2} = \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2} \cdot \left[\frac{dv}{\left(\frac{1}{\sqrt{2}}\right)^2 - v^2} \right] = \frac{dx}{x}$$

Integrating both sides, we get:

$$\frac{1}{2} \cdot \frac{1}{2 \times \frac{1}{\sqrt{2}}} \log \left| \frac{\frac{1}{\sqrt{2}} + v}{\frac{1}{\sqrt{2}} - v} \right| = \log|x| + C$$

$$\Rightarrow \frac{1}{2\sqrt{2}} \log \left| \frac{\frac{1}{\sqrt{2}} + \frac{y}{x}}{\frac{1}{\sqrt{2}} - \frac{y}{x}} \right| = \log|x| + C$$

$$\Rightarrow \frac{1}{2\sqrt{2}} \log \left| \frac{x + \sqrt{2}y}{x - \sqrt{2}y} \right| = \log|x| + C$$

This is the required solution for the given differential equation.

Question 6:

$$x dy - y dx = \sqrt{x^2 + y^2} dx$$

Answer

$$x dy - y dx = \sqrt{x^2 + y^2} dx$$

$$\Rightarrow x dy = \left[y + \sqrt{x^2 + y^2} \right] dx$$

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x} \quad \dots(1)$$

$$\text{Let } F(x, y) = \frac{y + \sqrt{x^2 + y^2}}{x}.$$

$$\therefore F(\lambda x, \lambda y) = \frac{\lambda y + \sqrt{(\lambda x)^2 + (\lambda y)^2}}{\lambda x} = \frac{y + \sqrt{x^2 + y^2}}{x} = \lambda^0 \cdot F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$$y = vx$$

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

Substituting the values of v and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x \frac{dv}{dx} = \frac{vx + \sqrt{x^2 + (vx)^2}}{x}$$

$$\Rightarrow v + x \frac{dv}{dx} = v + \sqrt{1 + v^2}$$

$$\Rightarrow \frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}$$

Integrating both sides, we get:

$$\log \left| v + \sqrt{1 + v^2} \right| = \log |x| + \log C$$

$$\Rightarrow \log \left| \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} \right| = \log |Cx|$$

$$\Rightarrow \log \left| \frac{y + \sqrt{x^2 + y^2}}{x} \right| = \log |Cx|$$

$$\Rightarrow y + \sqrt{x^2 + y^2} = Cx^2$$

This is the required solution of the given differential equation.

Question 7:

$$\left\{ x \cos\left(\frac{y}{x}\right) + y \sin\left(\frac{y}{x}\right) \right\} y dx = \left\{ y \sin\left(\frac{y}{x}\right) - x \cos\left(\frac{y}{x}\right) \right\} x dy$$

Answer

The given differential equation is:

$$\left\{ x \cos\left(\frac{y}{x}\right) + y \sin\left(\frac{y}{x}\right) \right\} y dx = \left\{ y \sin\left(\frac{y}{x}\right) - x \cos\left(\frac{y}{x}\right) \right\} x dy$$

$$\frac{dy}{dx} = \frac{\left\{ x \cos\left(\frac{y}{x}\right) + y \sin\left(\frac{y}{x}\right) \right\} y}{\left\{ y \sin\left(\frac{y}{x}\right) - x \cos\left(\frac{y}{x}\right) \right\} x} \quad \dots(1)$$

$$\text{Let } F(x, y) = \frac{\left\{ x \cos\left(\frac{y}{x}\right) + y \sin\left(\frac{y}{x}\right) \right\} y}{\left\{ y \sin\left(\frac{y}{x}\right) - x \cos\left(\frac{y}{x}\right) \right\} x}.$$

$$\begin{aligned} \therefore F(\lambda x, \lambda y) &= \frac{\left\{ \lambda x \cos\left(\frac{\lambda y}{\lambda x}\right) + \lambda y \sin\left(\frac{\lambda y}{\lambda x}\right) \right\} \lambda y}{\left\{ \lambda y \sin\left(\frac{\lambda y}{\lambda x}\right) - \lambda x \cos\left(\frac{\lambda y}{\lambda x}\right) \right\} \lambda x} \\ &= \frac{\left\{ x \cos\left(\frac{y}{x}\right) + y \sin\left(\frac{y}{x}\right) \right\} y}{\left\{ y \sin\left(\frac{y}{x}\right) - x \cos\left(\frac{y}{x}\right) \right\} x} \\ &= \lambda^0 \cdot F(x, y) \end{aligned}$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$$y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x = \frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$\begin{aligned}
v+x \frac{dv}{dx} &= \frac{(x \cos v + vx \sin v) \cdot vx}{(vx \sin v - x \cos v) \cdot x} \\
\Rightarrow v+x \frac{dv}{dx} &= \frac{v \cos v + v^2 \sin v}{v \sin v - \cos v} \\
\Rightarrow x \frac{dv}{dx} &= \frac{v \cos v + v^2 \sin v}{v \sin v - \cos v} - v \\
\Rightarrow x \frac{dv}{dx} &= \frac{v \cos v + v^2 \sin v - v^2 \sin v + v \cos v}{v \sin v - \cos v} \\
\Rightarrow x \frac{dv}{dx} &= \frac{2v \cos v}{v \sin v - \cos v} \\
\Rightarrow \left[\frac{v \sin v - \cos v}{v \cos v} \right] dv &= \frac{2dx}{x} \\
\Rightarrow \left(\tan v - \frac{1}{v} \right) dv &= \frac{2dx}{x}
\end{aligned}$$

Integrating both sides, we get:

$$\log(\sec v) - \log v = 2 \log x + \log C$$

$$\Rightarrow \log\left(\frac{\sec v}{v}\right) = \log(Cx^2)$$

$$\Rightarrow \left(\frac{\sec v}{v}\right) = Cx^2$$

$$\Rightarrow \sec v = Cx^2 v$$

$$\Rightarrow \sec\left(\frac{y}{x}\right) = C \cdot x^2 \cdot \frac{y}{x}$$

$$\Rightarrow \sec\left(\frac{y}{x}\right) = Cxy$$

$$\Rightarrow \cos\left(\frac{y}{x}\right) = \frac{1}{Cxy} = \frac{1}{C} \cdot \frac{1}{xy}$$

$$\Rightarrow xy \cos\left(\frac{y}{x}\right) = k \quad \left(k = \frac{1}{C}\right)$$

This is the required solution of the given differential equation.

Question 8:

$$x \frac{dy}{dx} - y + x \sin\left(\frac{y}{x}\right) = 0$$

Answer

$$x \frac{dy}{dx} - y + x \sin\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow x \frac{dy}{dx} = y - x \sin\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y - x \sin\left(\frac{y}{x}\right)}{x} \quad \dots(1)$$

$$\text{Let } F(x, y) = \frac{y - x \sin\left(\frac{y}{x}\right)}{x}.$$

$$\therefore F(\lambda x, \lambda y) = \frac{\lambda y - \lambda x \sin\left(\frac{\lambda y}{\lambda x}\right)}{\lambda x} = \frac{y - x \sin\left(\frac{y}{x}\right)}{x} = \lambda^0 \cdot F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$$y = vx$$

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x \frac{dv}{dx} = \frac{vx - x \sin v}{x}$$

$$\Rightarrow v + x \frac{dv}{dx} = v - \sin v$$

$$\Rightarrow -\frac{dv}{\sin v} = \frac{dx}{x}$$

$$\Rightarrow \operatorname{cosec} v \, dv = -\frac{dx}{x}$$

Integrating both sides, we get:

$$\log |\operatorname{cosec} v - \cot v| = -\log x + \log C = \log \frac{C}{x}$$

$$\Rightarrow \operatorname{cosec} \left(\frac{y}{x} \right) - \cot \left(\frac{y}{x} \right) = \frac{C}{x}$$

$$\Rightarrow \frac{1}{\sin \left(\frac{y}{x} \right)} - \frac{\cos \left(\frac{y}{x} \right)}{\sin \left(\frac{y}{x} \right)} = \frac{C}{x}$$

$$\Rightarrow x \left[1 - \cos \left(\frac{y}{x} \right) \right] = C \sin \left(\frac{y}{x} \right)$$

This is the required solution of the given differential equation.

Question 9:

$$y dx + x \log \left(\frac{y}{x} \right) dy - 2x dy = 0$$

Answer

$$y dx + x \log \left(\frac{y}{x} \right) dy - 2x dy = 0$$

$$\Rightarrow y dx = \left[2x - x \log \left(\frac{y}{x} \right) \right] dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2x - x \log \left(\frac{y}{x} \right)} \quad \dots(1)$$

$$\text{Let } F(x, y) = \frac{y}{2x - x \log \left(\frac{y}{x} \right)}$$

$$\therefore F(\lambda x, \lambda y) = \frac{\lambda y}{2(\lambda x) - (\lambda x) \log \left(\frac{\lambda y}{\lambda x} \right)} = \frac{y}{2x - \log \left(\frac{y}{x} \right)} = \lambda^0 \cdot F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$$y = vx$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x \frac{dv}{dx} = \frac{vx}{2x - x \log v}$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{v}{2 - \log v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v}{2 - \log v} - v$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v - 2v + v \log v}{2 - \log v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v \log v - v}{2 - \log v}$$

$$\Rightarrow \frac{2 - \log v}{v(\log v - 1)} dv = \frac{dx}{x}$$

$$\Rightarrow \left[\frac{1 + (1 - \log v)}{v(\log v - 1)} \right] dv = \frac{dx}{x}$$

$$\Rightarrow \left[\frac{1}{v(\log v - 1)} - \frac{1}{v} \right] dv = \frac{dx}{x}$$

Integrating both sides, we get:

$$\int \frac{1}{v(\log v - 1)} dv - \int \frac{1}{v} dv = \int \frac{1}{x} dx$$

$$\Rightarrow \int \frac{dv}{v(\log v - 1)} - \log v = \log x + \log C \quad \dots(2)$$

$$\Rightarrow \text{Let } \log v - 1 = t$$

$$\Rightarrow \frac{d}{dv}(\log v - 1) = \frac{dt}{dv}$$

$$\Rightarrow \frac{1}{v} = \frac{dt}{dv}$$

$$\Rightarrow \frac{dv}{v} = dt$$

Therefore, equation (1) becomes:

$$\Rightarrow \int \frac{dt}{t} - \log v = \log x + \log C$$

$$\Rightarrow \log t - \log \left(\frac{y}{x} \right) = \log(Cx)$$

$$\Rightarrow \log \left[\log \left(\frac{y}{x} \right) - 1 \right] - \log \left(\frac{y}{x} \right) = \log(Cx)$$

$$\Rightarrow \log \left[\frac{\log \left(\frac{y}{x} \right) - 1}{\frac{y}{x}} \right] = \log(Cx)$$

$$\Rightarrow \frac{x}{y} \left[\log \left(\frac{y}{x} \right) - 1 \right] = Cx$$

$$\Rightarrow \log \left(\frac{y}{x} \right) - 1 = Cy$$

This is the required solution of the given differential equation.

Question 10:

$$\left(1 + e^{\frac{x}{y}} \right) dx + e^{\frac{x}{y}} \left(1 - \frac{x}{y} \right) dy = 0$$

Answer

$$\left(1 + e^{\frac{x}{y}} \right) dx + e^{\frac{x}{y}} \left(1 - \frac{x}{y} \right) dy = 0$$

$$\Rightarrow \left(1 + e^{\frac{x}{y}} \right) dx = -e^{\frac{x}{y}} \left(1 - \frac{x}{y} \right) dy$$

$$\Rightarrow \frac{dx}{dy} = \frac{-e^{\frac{x}{y}} \left(1 - \frac{x}{y}\right)}{1 + e^{\frac{x}{y}}} \quad \dots(1)$$

$$\text{Let } F(x, y) = \frac{-e^{\frac{x}{y}} \left(1 - \frac{x}{y}\right)}{1 + e^{\frac{x}{y}}}.$$

$$\therefore F(\lambda x, \lambda y) = \frac{-e^{\frac{\lambda x}{\lambda y}} \left(1 - \frac{\lambda x}{\lambda y}\right)}{1 + e^{\frac{\lambda x}{\lambda y}}} = \frac{-e^{\frac{x}{y}} \left(1 - \frac{x}{y}\right)}{1 + e^{\frac{x}{y}}} = \lambda^0 \cdot F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$$x = vy$$

$$\Rightarrow \frac{d}{dy}(x) = \frac{d}{dy}(vy)$$

$$\Rightarrow \frac{dx}{dy} = v + y \frac{dv}{dy}$$

Substituting the values of x and $\frac{dx}{dy}$ in equation (1), we get:

$$v + y \frac{dv}{dy} = \frac{-e^v(1-v)}{1+e^v}$$

$$\Rightarrow y \frac{dv}{dy} = \frac{-e^v + ve^v}{1+e^v} - v$$

$$\Rightarrow y \frac{dv}{dy} = \frac{-e^v + ve^v - v - ve^v}{1+e^v}$$

$$\Rightarrow y \frac{dv}{dy} = - \left[\frac{v + e^v}{1 + e^v} \right]$$

$$\Rightarrow \left[\frac{1 + e^v}{v + e^v} \right] dv = - \frac{dy}{y}$$

Integrating both sides, we get:

$$\Rightarrow \log(v + e^y) = -\log y + \log C = \log\left(\frac{C}{y}\right)$$

$$\Rightarrow \left[\frac{x}{y} + e^y\right] = \frac{C}{y}$$

$$\Rightarrow x + ye^y = C$$

This is the required solution of the given differential equation.

Question 11:

$$(x + y)dy + (x - y)dx = 0; y = 1 \text{ when } x = 1$$

Answer

$$(x + y)dy + (x - y)dx = 0$$

$$\Rightarrow (x + y)dy = -(x - y)dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(x - y)}{x + y} \quad \dots(1)$$

$$\text{Let } F(x, y) = \frac{-(x - y)}{x + y}.$$

$$\therefore F(\lambda x, \lambda y) = \frac{-(\lambda x - \lambda y)}{\lambda x + \lambda y} = \frac{-(x - y)}{x + y} = \lambda^0 \cdot F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$$y = vx$$

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$\begin{aligned}
 v+x \frac{dv}{dx} &= \frac{-(x-vx)}{x+vx} \\
 \Rightarrow v+x \frac{dv}{dx} &= \frac{v-1}{v+1} \\
 \Rightarrow x \frac{dv}{dx} &= \frac{v-1}{v+1} - v = \frac{v-1-v(v+1)}{v+1} \\
 \Rightarrow x \frac{dv}{dx} &= \frac{v-1-v^2-v}{v+1} = \frac{-(1+v^2)}{v+1} \\
 \Rightarrow \frac{(v+1)}{1+v^2} dv &= -\frac{dx}{x} \\
 \Rightarrow \left[\frac{v}{1+v^2} + \frac{1}{1+v^2} \right] dv &= -\frac{dx}{x}
 \end{aligned}$$

Integrating both sides, we get:

$$\begin{aligned}
 \frac{1}{2} \log(1+v^2) + \tan^{-1} v &= -\log x + k \\
 \Rightarrow \log(1+v^2) + 2 \tan^{-1} v &= -2 \log x + 2k \\
 \Rightarrow \log \left[(1+v^2) \cdot x^2 \right] + 2 \tan^{-1} v &= 2k \\
 \Rightarrow \log \left[\left(1 + \frac{y^2}{x^2} \right) \cdot x^2 \right] + 2 \tan^{-1} \frac{y}{x} &= 2k \\
 \Rightarrow \log(x^2 + y^2) + 2 \tan^{-1} \frac{y}{x} &= 2k \quad \dots(2)
 \end{aligned}$$

Now, $y = 1$ at $x = 1$.

$$\begin{aligned}
 \Rightarrow \log 2 + 2 \tan^{-1} 1 &= 2k \\
 \Rightarrow \log 2 + 2 \times \frac{\pi}{4} &= 2k \\
 \Rightarrow \frac{\pi}{2} + \log 2 &= 2k
 \end{aligned}$$

Substituting the value of $2k$ in equation (2), we get:

$$\log(x^2 + y^2) + 2 \tan^{-1} \left(\frac{y}{x} \right) = \frac{\pi}{2} + \log 2$$

This is the required solution of the given differential equation.

Question 12:

$$x^2 dy + (xy + y^2) dx = 0; y = 1 \text{ when } x = 1$$

Answer

$$x^2 dy + (xy + y^2) dx = 0$$

$$\Rightarrow x^2 dy = -(xy + y^2) dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(xy + y^2)}{x^2} \quad \dots(1)$$

$$\text{Let } F(x, y) = \frac{-(xy + y^2)}{x^2}.$$

$$\therefore F(\lambda x, \lambda y) = \frac{[\lambda x \cdot \lambda y + (\lambda y)^2]}{(\lambda x)^2} = \frac{-(xy + y^2)}{x^2} = \lambda^0 \cdot F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$$y = vx$$

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x \frac{dv}{dx} = \frac{-[x \cdot vx + (vx)^2]}{x^2} = -v - v^2$$

$$\Rightarrow x \frac{dv}{dx} = -v^2 - 2v = -v(v + 2)$$

$$\Rightarrow \frac{dv}{v(v + 2)} = -\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2} \left[\frac{(v + 2) - v}{v(v + 2)} \right] dv = -\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2} \left[\frac{1}{v} - \frac{1}{v + 2} \right] dv = -\frac{dx}{x}$$

Integrating both sides, we get:

$$\frac{1}{2}[\log v - \log(v+2)] = -\log x + \log C$$

$$\Rightarrow \frac{1}{2} \log\left(\frac{v}{v+2}\right) = \log \frac{C}{x}$$

$$\Rightarrow \frac{v}{v+2} = \left(\frac{C}{x}\right)^2$$

$$\Rightarrow \frac{\frac{y}{x}}{\frac{y}{x}+2} = \left(\frac{C}{x}\right)^2$$

$$\Rightarrow \frac{y}{y+2x} = \frac{C^2}{x^2}$$

$$\Rightarrow \frac{x^2 y}{y+2x} = C^2 \quad \dots(2)$$

Now, $y = 1$ at $x = 1$.

$$\Rightarrow \frac{1}{1+2} = C^2$$

$$\Rightarrow C^2 = \frac{1}{3}$$

Substituting $C^2 = \frac{1}{3}$ in equation (2), we get:

$$\frac{x^2 y}{y+2x} = \frac{1}{3}$$

$$\Rightarrow y+2x = 3x^2 y$$

This is the required solution of the given differential equation.

Question 13:

$$\left[x \sin^2\left(\frac{x}{y} - y\right) \right] dx + x dy = 0; y = \frac{\pi}{4} \text{ when } x = 1$$

Answer

$$\left[x \sin^2 \left(\frac{y}{x} \right) - y \right] dx + x dy = 0$$

$$\Rightarrow \frac{dy}{dx} = - \frac{\left[x \sin^2 \left(\frac{y}{x} \right) - y \right]}{x} \quad \dots(1)$$

$$\text{Let } F(x, y) = \frac{- \left[x \sin^2 \left(\frac{y}{x} \right) - y \right]}{x}$$

$$\therefore F(\lambda x, \lambda y) = \frac{- \left[\lambda x \cdot \sin^2 \left(\frac{\lambda y}{\lambda x} \right) - \lambda y \right]}{\lambda x} = \frac{- \left[x \sin^2 \left(\frac{y}{x} \right) - y \right]}{x} = \lambda^0 \cdot F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve this differential equation, we make the substitution as:

$$y = vx$$

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x \frac{dv}{dx} = \frac{- \left[x \sin^2 v - vx \right]}{x}$$

$$\Rightarrow v + x \frac{dv}{dx} = - \left[\sin^2 v - v \right] = v - \sin^2 v$$

$$\Rightarrow x \frac{dv}{dx} = - \sin^2 v$$

$$\Rightarrow \frac{dv}{\sin^2 v} = - \frac{dx}{x}$$

$$\Rightarrow \operatorname{cosec}^2 v dv = - \frac{dx}{x}$$

Integrating both sides, we get:

$$-\cot v = -\log|x| - C$$

$$\Rightarrow \cot v = \log|x| + C$$

$$\Rightarrow \cot\left(\frac{y}{x}\right) = \log|x| + \log C$$

$$\Rightarrow \cot\left(\frac{y}{x}\right) = \log|Cx| \quad \dots(2)$$

Now, $y = \frac{\pi}{4}$ at $x = 1$

$$\Rightarrow \cot\left(\frac{\pi}{4}\right) = \log|C|$$

$$\Rightarrow 1 = \log C$$

$$\Rightarrow C = e^1 = e$$

Substituting $C = e$ in equation (2), we get:

$$\cot\left(\frac{y}{x}\right) = \log|ex|$$

This is the required solution of the given differential equation.

Question 14:

$$\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\left(\frac{y}{x}\right) = 0; y = 0 \text{ when } x = 1$$

Answer

$$\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \operatorname{cosec}\left(\frac{y}{x}\right) \quad \dots(1)$$

$$\text{Let } F(x, y) = \frac{y}{x} - \operatorname{cosec}\left(\frac{y}{x}\right).$$

$$\therefore F(\lambda x, \lambda y) = \frac{\lambda y}{\lambda x} - \operatorname{cosec}\left(\frac{\lambda y}{\lambda x}\right)$$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{y}{x} - \operatorname{cosec}\left(\frac{y}{x}\right) = F(x, y) = \lambda^0 \cdot F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$$y = vx$$

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x \frac{dv}{dx} = v - \operatorname{cosec} v$$

$$\Rightarrow -\frac{dv}{\operatorname{cosec} v} = -\frac{dx}{x}$$

$$\Rightarrow -\sin v dv = \frac{dx}{x}$$

Integrating both sides, we get:

$$\cos v = \log x + \log C = \log |Cx|$$

$$\Rightarrow \cos\left(\frac{y}{x}\right) = \log |Cx| \quad \dots(2)$$

This is the required solution of the given differential equation.

Now, $y = 0$ at $x = 1$.

$$\Rightarrow \cos(0) = \log C$$

$$\Rightarrow 1 = \log C$$

$$\Rightarrow C = e^1 = e$$

Substituting $C = e$ in equation (2), we get:

$$\cos\left(\frac{y}{x}\right) = \log |ex|$$

This is the required solution of the given differential equation.

Question 15:

$$2xy + y^2 - 2x^2 \frac{dy}{dx} = 0; y = 2 \text{ when } x = 1$$

Answer

$$2xy + y^2 - 2x^2 \frac{dy}{dx} = 0$$

$$\Rightarrow 2x^2 \frac{dy}{dx} = 2xy + y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy + y^2}{2x^2} \quad \dots(1)$$

$$\text{Let } F(x, y) = \frac{2xy + y^2}{2x^2}.$$

$$\therefore F(\lambda x, \lambda y) = \frac{2(\lambda x)(\lambda y) + (\lambda y)^2}{2(\lambda x)^2} = \frac{2xy + y^2}{2x^2} = \lambda^0 \cdot F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$$y = vx$$

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

Substituting the value of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x \frac{dv}{dx} = \frac{2x(vx) + (vx)^2}{2x^2}$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{2v + v^2}{2}$$

$$\Rightarrow v + x \frac{dv}{dx} = v + \frac{v^2}{2}$$

$$\Rightarrow \frac{2}{v^2} dv = \frac{dx}{x}$$

Integrating both sides, we get:

$$\begin{aligned}2 \cdot \frac{v^{-2+1}}{-2+1} &= \log|x| + C \\ \Rightarrow -\frac{2}{v} &= \log|x| + C \\ \Rightarrow -\frac{2}{\frac{y}{x}} &= \log|x| + C \\ \Rightarrow -\frac{2x}{y} &= \log|x| + C \quad \dots(2)\end{aligned}$$

Now, $y = 2$ at $x = 1$.

$$\Rightarrow -1 = \log(1) + C$$

$$\Rightarrow C = -1$$

Substituting $C = -1$ in equation (2), we get:

$$\begin{aligned}-\frac{2x}{y} &= \log|x| - 1 \\ \Rightarrow \frac{2x}{y} &= 1 - \log|x| \\ \Rightarrow y &= \frac{2x}{1 - \log|x|}, \quad (x \neq 0, x \neq e)\end{aligned}$$

This is the required solution of the given differential equation.

Question 16:

A homogeneous differential equation of the form $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$ can be solved by making the substitution

A. $y = vx$

B. $v = yx$

C. $x = vy$

D. $x = v$

Answer

For solving the homogeneous equation of the form $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$, we need to make the substitution as $x = vy$.

Hence, the correct answer is C.

Question 17:

Which of the following is a homogeneous differential equation?

A. $(4x + 6y + 5)dy - (3y + 2x + 4)dx = 0$

B. $(xy)dx - (x^3 + y^3)dy = 0$

C. $(x^3 + 2y^2)dx + 2xy dy = 0$

D. $y^2 dx + (x^2 - xy^2 - y^2)dy = 0$

Answer

Function $F(x, y)$ is said to be the homogenous function of degree n , if

$F(\lambda x, \lambda y) = \lambda^n F(x, y)$ for any non-zero constant (λ) .

Consider the equation given in alternative D:

$$y^2 dx + (x^2 - xy - y^2)dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y^2}{x^2 - xy - y^2} = \frac{y^2}{y^2 + xy - x^2}$$

$$\text{Let } F(x, y) = \frac{y^2}{y^2 + xy - x^2}.$$

$$\begin{aligned} \Rightarrow F(\lambda x, \lambda y) &= \frac{(\lambda y)^2}{(\lambda y)^2 + (\lambda x)(\lambda y) - (\lambda x)^2} \\ &= \frac{\lambda^2 y^2}{\lambda^2 (y^2 + xy - x^2)} \\ &= \lambda^0 \left(\frac{y^2}{y^2 + xy - x^2} \right) \\ &= \lambda^0 \cdot F(x, y) \end{aligned}$$

Hence, the differential equation given in alternative **D** is a homogenous equation.